PARTICLE KINEMATICS

1. INTRODUCTION

Motion is common to everything in the universe. We walk, run and ride a bicycle. Even when we are sleeping, air moves into and out of our lungs and blood flows in arteries and veins. The study of motion of objects along a straight line, also known as rectilinear motion. In Kinematics, we study ways to describe motion without going into the causes of motion. Particle kinematics deals with nature of motion i.e. how fast and on what path an object moves and relates the position, velocity, acceleration, and time without any reference to mass, force and energy. In other words, it is study of geometry of motion.

1.1 PARTICLE

A body that have a definite size is considered as a particle when all of its parts are constraint to move together. The motion of any such body can be studied by the motion of any point on that body. Hence, a particle is considered as equivalent to a point.

1.2 MOTION

When the position of a particle changes with time, it is said to be in motion with respect to the observer. For example, a person standing on ground observes that the train is moving with respect to him, while for a person in train, the train is at rest. There are two types of motions, translational and rotational. We will discuss translational motion in detail in this unit. A body in translation motion can move on either a straight–line path or curvilinear path.

2. TYPES OF TRANSLATION MOTION

2.1 RECTILINEAR MOTION

Motion on a straight–line path is known as rectilinear translation motion. It is also known as one–dimensional motion. A car running on a straight road, train running on a straight track and a ball thrown vertically upwards or dropped from a height etc are very common examples of rectilinear translation motion.

2.2 CURVILINEAR MOTION

(a) Two Dimensional Motion - Motion of a body on a curve path is known as curvilinear translation motion. If the path traced by the particle is in a plane, the motion is known as two–dimensional motion. A ball hit by a batsman at some angle with the horizontal describes a curvilinear trajectory in a vertical plane and the hands of a clock describes a circular path are examples of two-dimensional motion.

(b) Three Dimensional Motion - If path is not in a plane and requires a region of space or volume, the motion is known as three–dimensional motion or motion is space. An insect flying randomly in a room, motion of a football in soccer game over considerable duration of time etc are common examples of three–dimensional motion.

FRAME OF REFERENCE

Motion is change in position of an object with time. In order to specify position, we need to use a reference point and a set of axes. It is convenient to choose a rectangular coordinate system consisting of three mutually perpendicular axes, labelled X, Y, and Z - axes. The point of intersection of these three axes is called origin (O) and serves as the reference point. The coordinates (x, y. z) of an object describe the position of the object with respect to this coordinate system. To measure time, we position a clock in this system. This coordinate system along with a clock constitutes a frame of reference.

If one or more coordinates of an object change with time, we say that the object is in motion. Otherwise, the object is said to be at rest with respect to this frame of reference.

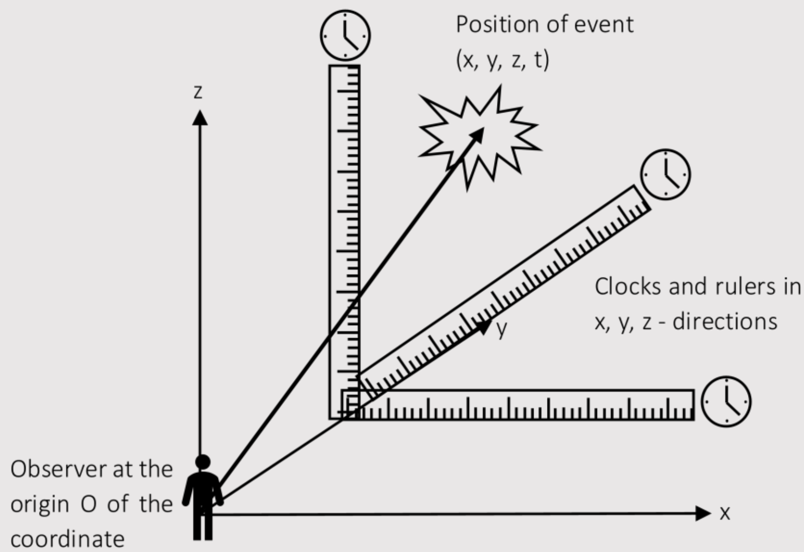
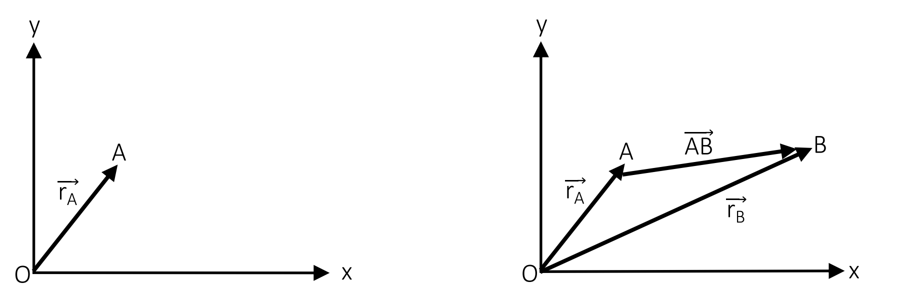


Figure 1: Frame of Reference

3. POSITION AND POSITION VECTORS

Position of any particle can be represented by its coordinates with respect to an origin in Cartesian system. Position vector describes position of a particle relative to another particle and is a vector from the later towards the first. To study motion of a particle we have to assume a reference frame fixed with some other body. Suppose there are two particles A and B. The vector drawn from the origin of the coordinate system representing the reference frame to the location of the particle A is known as position vector of the particle A with respect to the reference frame. The vector $\vec{r}_A$ is the position vector of the particle A as shown in the figure.



(a) Position vector of the particle A (b) Position vector of the particle B w.r.t A

Figure 2: Position Vectors

Position vector of particle A in three dimensions: $\vec{r}_A = \overrightarrow{OA} = x_A\hat{i} + y_A\hat{j} + z_A\hat{k}$

Similarly, position vector of particle B in three dimensions: $\vec{r_B} = \overrightarrow{OB} = x_B\hat{i} + y_B\hat{j} + z_B\hat{k}$

The position vector of B with respect to A can be determined as: $\overrightarrow{AB} = \vec{r_B} - \vec{r_A} = (x_B - x_A)\hat{i} + (y_B - y_A)\hat{j} + (z_B - z_A)\hat{k}$

Illustration 1: Find the torque (τ) exerted by force $\hat{i} + 4\hat{j} + 3\hat{k}$ at point P (1, 3, 6) with respect to origin. Given, $\vec{\tau} = \vec{r} \times \vec{F}$. **[JEE MAIN]**

Sol: $\vec{r} = \hat{i} + 3\hat{j} + 6\hat{k}$ (Given)

$\vec{F} = \hat{i} + 4\hat{j} + 3\hat{k}$ (Given)

We know, $\vec{\tau} = \vec{r} \times \vec{F}$

$$\therefore \quad \vec{\tau} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 6 \\ 1 & 4 & 3 \end{vmatrix} = -15\hat{i} + 3\hat{j} + 1\hat{k} = -15\hat{i} + 3\hat{j} + \hat{k}$$

4. DISTANCE AND DISPLACEMENT

When a particle move in space it either follow a curve or a straight line. This curve or line traced by the particle is known as its trajectory. Let the particle moves from point P to R on the curvilinear path as shown in the figure.

The distance travelled by the particle is length of the path or curve traversed by it. It is also called “path length”. In the figure, the length of the curve PQR (the dashed line) is the distance travelled by the particle. It is a scalar quantity. SI unit of distance is metre.

The minimum distance between the points P and R is the displacement. It is a vector quantity. The vector $\overrightarrow{PR} = \vec{r}_R - \vec{r}_P$ is the displacement vector. The SI unit of displacement is same as that of distance, metre.

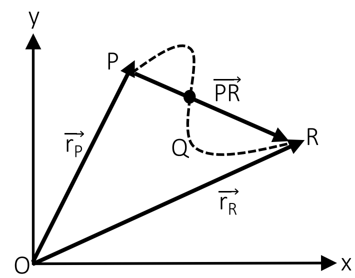


Figure 3: Distance and Displacement

NOTE

Distance traveled between two places is greater than the magnitude of displacement vector wherever particle changes its direction during its motion. In unidirectional motion, both of them are equal.

Illustration 2: Find the distance and displacement of a particle travelling from one point to another, say from point A to B, in the path given in the figure. **[JEE MAIN]**

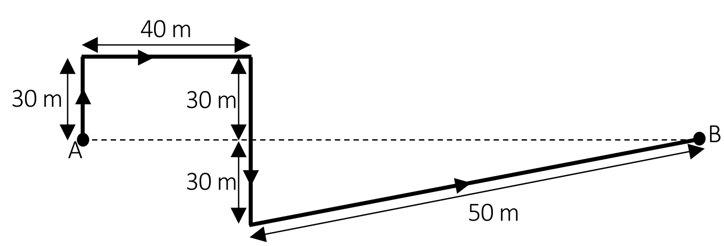


Figure 4

Sol: Total distance travelled is the path traversed by the particle from point A to B. Displacement is the shortest distance between the points A and B.

Total Distance Travelled = 30 + 40 + 30 + 30 + 50 = 180 m

Total Displacement = $40 + \sqrt{50^2 - 30^2} = 40 + 40 = 80$ m

5. VELOCITY, SPEED AND ACCELERATION

5.1 AVERAGE VELOCITY AND AVERAGE SPEED

Average velocity is defined as the change in position or displacement '$\Delta\vec{r}$' divided by the time intervals 'Δt', in which the displacement occurs. It is defined as the ratio of displacement to the concerned time interval.

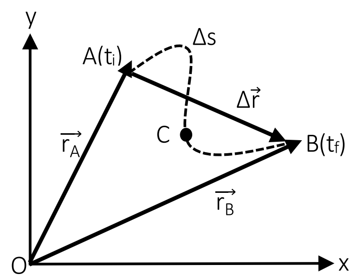


Figure 5: Average Speed and Average Velocity

Suppose the particle moves from point A to point B via point C in time interval t_i to t_f, as shown in the figure, the average velocity $\vec{V}_{avg}$ in this time interval is given by:

$$\vec{V}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}_f - \vec{r}_i}{t_f - t_i}$$

Average speed is defined as the total path length travelled divided by the total time interval during which the motion has taken place. It is defined as the ratio of distance travelled to the concerned time interval. The particle travels path length i.e., the length of the path ACB (equal to Δs) in time interval t_i to t_f, its average speed is given by the following equation.

$$\text{Average Speed} = \frac{\text{Total Path length}}{\text{Total Time Interval}} = \frac{\Delta s}{\Delta t} = \frac{\text{Total Path length}}{t_f - t_i}$$

SI unit of both velocity and speed is ms^{-1}. The speed is a scalar quantity whereas velocity is a vector quantity.

NOTE

Average speed in a time interval is greater than the magnitude of average velocity vector wherever particle changes its direction during its motion. In unidirectional motion, both of them are equal.

Illustration 3: If a car moves from point A to B with a constant speed $v_1 = 50$ km h^{-1} and returns back to the initial point A with a constant speed $v_2 = 70$ km h^{-1}, then find the average speed and average velocity. **[JEE MAIN]**

Sol: Let the distance AB = s

Time taken by car from A to B, $t_1 = \frac{s}{v_1}$

Time taken by car From B to A, $t_2 = \frac{s}{v_2}$

$$\text{Average speed} = \frac{\text{Total distance}}{\text{Total time taken}} = \frac{s+s}{t_1+t_2} = \frac{s+s}{\frac{s}{v_1}+\frac{s}{v_2}} = \frac{2v_1v_2}{v_1+v_2} = \frac{2\times 50\times 70}{50+70} = \frac{7000}{120} = 58.33 \text{ km h}^{-1}$$

$$\text{Average velocity, } \vec{v}_{avg} = \frac{\text{Displacement}}{\text{Total time}} = \frac{0}{t_1+t_2} = 0$$

Illustration 4: A particle travels half of the journey with speed 2 ms^{-1}. For second half of the journey, the particle travels with a speed of 3 ms^{-1} for half of remaining time, and for the other half it travel with a speed of 6 ms^{-1}. Find its average speed. **[JEE ADVANCED]**

Sol: Average speed is distance covered divided by total time taken. Here we need to assume total distance of journey. The speed in each part of the journey is constant. The time taken to cover each part of the journey can be calculated in terms of distance covered and speed in that part.

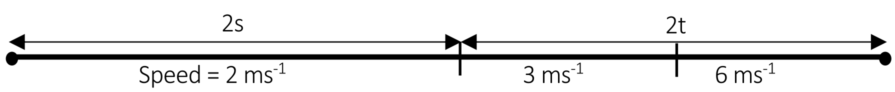


Figure 6

Let total distance = 4s

Let the time taken in covering last half of journey = 2t

For the first half of the journey

Speed, $v_1 = 2$ ms^{-1} and Distance, $d_1 = 2s$

$$\text{So, time taken, } t_1 = \frac{\text{Distance}}{\text{Speed}} = \frac{2s}{2} = s$$

For the first part of second half of the journey

$v_2 = 3$ ms^{-1}, $t_2 = t$ s,

$\therefore \quad d_2 = v_2 \times t_2 = 3t$

For the second part of second half of the journey

$v_3 = 6$ ms^{-1}, $t_3 = t$ s, $d_3 = v_3 \times t_3 = 6t$

$$\text{We know } d_2 + d_3 = \frac{\text{Total distance}}{2} = \frac{4s}{2} = 2s$$

$\Rightarrow$ $3t + 6t = 2s$

$$\Rightarrow \quad t = \frac{2s}{9}$$

$$\text{Therefore, total time taken for journey} = t_1 + t_2 + t_3 = s + t + t = s + \frac{2s}{9} + \frac{2s}{9} = \frac{13s}{9}$$

$$\therefore \quad \text{Average Speed} = \frac{\text{Total distance}}{\text{Total time}} = \frac{4s}{\frac{13s}{9}} = \frac{36}{13} \text{ ms}^{-1}$$

Illustration 5: Men running in a straight line with a speed of 15 ms^{-1} one behind the other at equal intervals of 20 m. Cyclists are also riding along the straight line with a speed of 25 ms^{-1} at equal intervals of 30 m. An observer moving in the opposite direction with a speed of v such that comes across men and cyclists at the same time. Find the velocity v. **[JEE ADVANCED]**

Sol: The relative velocity between observer and men is (15 + v). The relative velocity between observer and cyclists is (25 + v). In the reference frame of observer, the time elapsed between the passing of two men is 20/(15 + v) and the time elapsed between the passing of two cyclists is 30/(25 + v). Both these timings should be equal so that the men and cyclists pass the observer together.

Let us assume that a man, a cyclist and the observer are in line. Now after t time, the observer again meets with the next man and cyclist.

Then the distance travelled by the observer = vt

Distance travelled by the man = 15t = 20 – vt ... (i)

Distance travelled by the cyclist = 25t = 30 – vt ... (ii)

Simplifying (i) and (ii) we get

15t + vt = 20 ... (iii)

25t + vt = 30 ... (iv)

Dividing (iii) and (iv) we get, $\frac{15t + vt}{25t + vt} = \frac{20}{30}$

$\Rightarrow \quad \frac{15 + v}{25 + v} = \frac{2}{3}$

$\Rightarrow \quad 45 + 3v = 50 + 2v$

$\Rightarrow \quad v = 5\ ms^{-1}$

Illustration 6: Two Cars A and B simultaneously start with speed 20 ms^{-1} and 30 ms^{-1}, respectively. Both cars have constant but different accelerations. On completing the race simultaneously, if the final velocity of A is 90 ms^{-1}, then find the final velocity of B.

[JEE ADVANCED]

Sol: Both cars travel equal distance in equal time. Initial velocities and accelerations of both the cars are different. Problem can be best solved by equating the average velocities.

Since both cars travel equal distance at equal intervals of time, both cars have equal average speeds.

For constant acceleration, $v_{avg} = \frac{u + v}{2}$

$\therefore \quad \frac{20 + 90}{2} = \frac{30 + v_B}{2}$

$\Rightarrow \quad v_B = 80\ ms^{-1}$

5.2 INSTANTANEOUS VELOCITY AND INSTANTANEOUS SPEED

The average velocity of a moving body with infinitesimally small time interval is known as instantaneous velocity. Assume that the time interval Δt to be infinitesimally small (i.e. Δt→0), the point B approaches A making the chord AB to coincide with the tangent at A (figure 5 becomes figure 7).

The instantaneous velocity equals to the rate of change in its position vector $\vec{r}$ with time. Its direction is along the tangent to the path. The instantaneous velocity is given by the following equation.

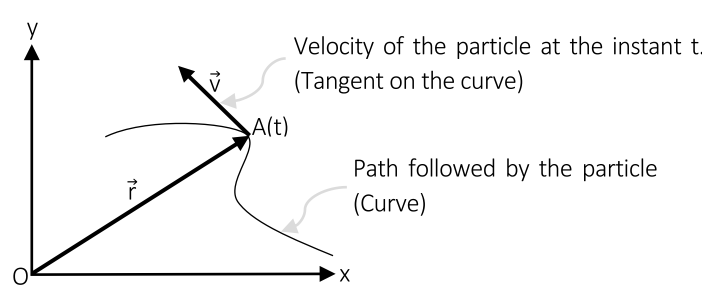


Figure 7: Instantaneous Velocity

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

Instantaneous speed is defined as the time rate of distance travelled. Instantaneous speed is given by the following equation

$$\text{Speed} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

You can easily conceive that when $\Delta t \to 0$, not only the chord AB but also the arc ACB (see figure 4) both approach to coincide with each other and with the tangent. Therefore $ds = |d\vec{r}|$. Now we can say that speed equals to magnitude of instantaneous velocity.

Instantaneous speed tells us how fast a particle moves at an instant and instantaneous velocity tells us in what direction and with what speed a particle moves at an instant of time.

Illustration 7: The distance travelled by a particle in time t is given by $s(t) = 2.5t^2$. Then find:

(a) The average speed of the particle during time 0 to 5 s?

(b) The instantaneous speed at t = 5.0 s **[JEE MAIN]**

Sol: Average speed is distance covered divided by total time taken. Instantaneous speed is the rate of change of distance at a particular instant.

(a) The distance travelled during time 0 to 5.0 s = $s(5) = 2.5 \times (5)^2 = 62.5$ m

$\therefore \quad \text{Average Speed} = \dfrac{\text{Distance Travelled}}{\text{Time Interval}}$

$\Rightarrow \quad \text{Average Speed} = \dfrac{62.5}{5 - 0} = 12.5\ ms^{-1}$

(b) $s(t) = 2.5t^2$

$\therefore \quad \text{Instantaneous Speed} = \dfrac{ds(t)}{dt} = 2.5 \times 2t = 5t = 5 \times 5 = 25\ ms^{-1}$

5.3 AVERAGE AND INSTANTANEOUS ACCELERATION

Acceleration is defined as the rate of change of velocity. For a particle having velocity $\vec{v}_1$ at time t_1 and $\vec{v}_2$ at time t_2, the average acceleration is defined by the following equation.

$$\vec{a}_{avg} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$$

Acceleration is a vector quantity and its SI unit is ms^{-2}.

Instantaneous acceleration is measure of how fast velocity of a body changes i.e. how fast direction of motion and speed change with time. The average acceleration with infinitesimally small time interval Δt becomes instantaneous acceleration. It is given by the following equation.

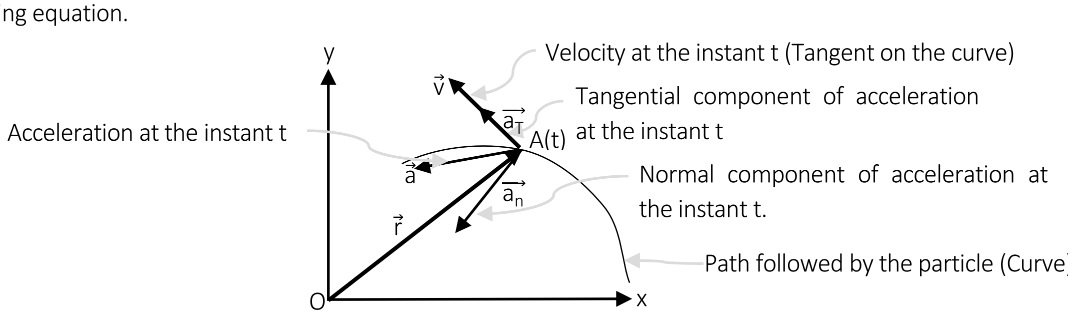


Figure 8: Acceleration

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

A vector quantity changes when its magnitude or direction or both changes. Accordingly, acceleration vector may have two components, one responsible to change only speed and the other responsible to change only direction of motion.

Component of acceleration responsible to change speed must be in the direction of motion. It is known as tangential component of acceleration $\vec{a}_T$. The component responsible to change direction of motion must be perpendicular to the direction of motion. It is known as normal component of acceleration $\vec{a}_n$. Acceleration vector $\vec{a}$ of a particle moving on a curvilinear path and its tangential and Normal components are shown in the figure.

Illustration 8: Consider a particle moving with a speed of 5 ms^{-1} towards east. After 10 sec velocity of particle is 5 ms^{-1} towards north. Find the average acceleration and its direction. **[JEE MAIN]**

Sol: Average acceleration is change in velocity divided by total time taken.

$\vec{v}_i = 5\hat{i}$ and $\vec{v}_f = 5\hat{j}$ which implies $\vec{v}_f - \vec{v}_i = 5\hat{j} - 5\hat{i}$

Times interval = 10 sec

We know that , Average acceleration, $\vec{a}_{avg} = \frac{\vec{v}_f - \vec{v}_i}{\text{Times interval}} = \frac{5\hat{j} - 5\hat{i}}{10} = \frac{1}{2}(\hat{j} - \hat{i})\ ms^{-2}$

$\therefore \quad \left|\vec{a}_{avg}\right| = \left|\frac{1}{2}(\hat{j} - \hat{i})\right| = \frac{1}{\sqrt{2}}(\hat{j} - \hat{i})\ ms^{-2}$

Unit vector along that direction is $= \frac{1}{\sqrt{2}}(\hat{j} - \hat{i})\ ms^{-2}$ [45° due west of north]

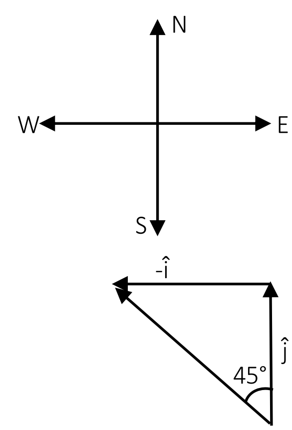


Figure 9

5.4 POSITION, VELOCITY AND ACCELERATION IN THREE DIMENSIONS

For a particle moving on a curvilinear path AB in three dimensions. Position of the particle at an instant of time t is at a point P (x, y, z) as shown in the figure. Let its velocity and acceleration at point P be $\hat{v}$ and acceleration $\hat{a}$. Its position vector is written as follows.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

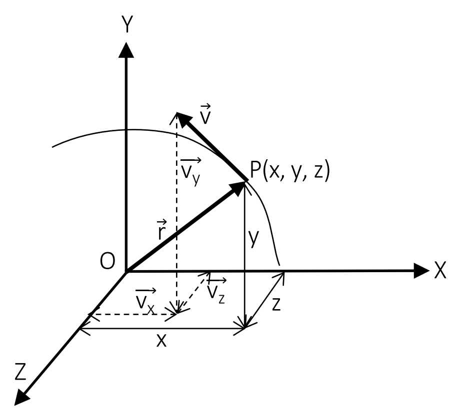


Figure 10: Velocity in three dimensions

Differentiating it with respect to time, we get velocity vector.

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

Here $v_x = \frac{dx}{dt}$, $v_y = \frac{dy}{dt}$, $v_z = \frac{dz}{dt}$ are the components of velocity vectors in the x, y and z directions respectively.

Now the acceleration can be obtained by differentiating velocity vector $\vec{v}$ with respect to time.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

Acceleration vector can also be obtained by differentiating position vector twice with respect to time.

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

In the above two equations, $a_x = \frac{d^2x}{dt^2} = \frac{dv_x}{dt}$, $a_y = \frac{d^2y}{dt^2} = \frac{dv_y}{dt}$ and $a_z = \frac{d^2z}{dt^2} = \frac{dv_z}{dt}$ are the components of acceleration vectors in the x, y and z directions respectively.

In the above equations, we can analyse each of the components x, y and z of motion as three individual rectilinear motions each along one of the axes x, y and z. A curvilinear motion can be analysed as superposition of three simultaneous rectilinear motions each along one of the coordinate axes.

6. RECTILINEAR MOTION

When the trajectory of the particle has only dimension then its motion is called rectilinear motion. Curvilinear motion can be conceived as superposition of three rectilinear motions each along one of the Cartesian axes.

6.1 UNIFORM VELOCITY MOTION (NO ACCELERATION)

In uniform velocity motion, a body moves with constant speed on a straight–line path without change in direction. For a body initially at $x = x_0$ at the instant $t = 0$, moves with uniform velocity v in the positive x–direction, its equation of motion at any time t is given as follows:

$$\boxed{x = x_0 + vt}$$

Figure 11 shows the velocity-time and position-time graph for the motion of the particle. The area under velocity-time graph and the time axes gives the change in position. Therefore, position–time relation or position at any instant can be determined.

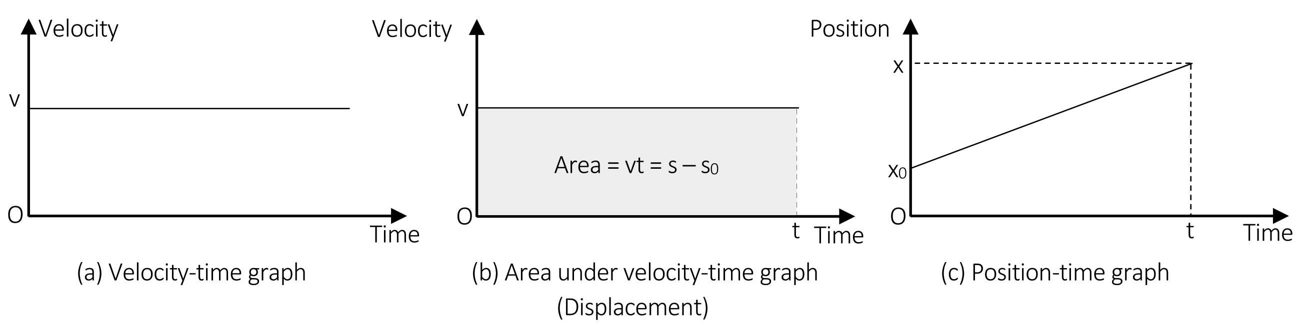
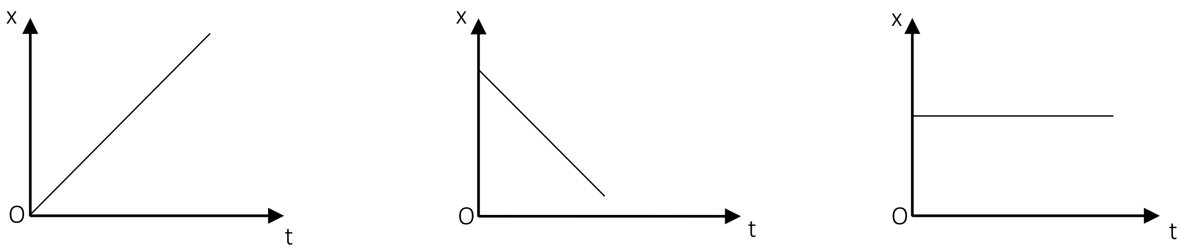


Figure 11: Velocity-time and position-time graphs for uniform velocity motion

Position-time graph for the particle having positive, negative and zero velocity are given in figure 12. The slope of position-time graph gives velocity of the particle and also the nature of the velocity.



(a) $v > 0$ (b) $v < 0$ (c) $v = 0$

Figure 12: Displacement time graphs for uniform velocity motion

Illustration 9: Displacement–time graph of a bus moving in a straight line is shown in the figure. State whether the motion is accelerated or not. Describe the motion of the bus in detail. Given $s_0 = 20$ m and $t_0 = 4$ s. **[JEE MAIN]**

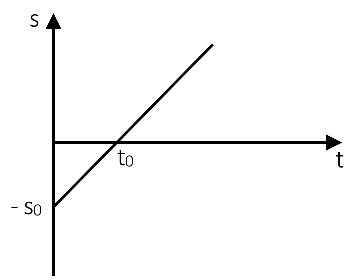


Figure 13: Displacement-time graph

Sol: The velocity of the bus at any instant is the slope of the displacement time graph at that instant. If the slope is constant, velocity is constant.

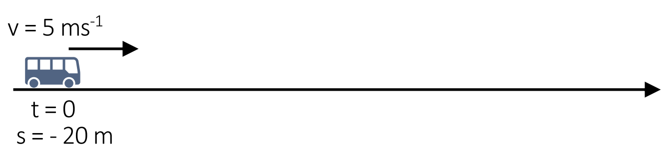


Figure 14

The slope of displacement-time graph is a straight line. Hence, velocity of particle is constant. At time t = 0, displacement of the bus from its mean position is $-s_0 = -20$ m. Velocity of bus,

$$v = \text{slope} = \frac{s_0}{t_0} = \frac{20}{4} = 5\ \text{ms}^{-1}$$

At t = 0 bus is at - 20 m and has a constant velocity of 5 ms^{-1}. At $t_0 = 4$ sec, bus will pass through its mean position.

6.2 UNIFORMLY ACCELERATED MOTION

Motion in which acceleration remains constant in magnitude as well as direction is called uniform acceleration motion. For this kind of motion, a general set of equations can be derived using calculus by taking acceleration (rate of change of velocity) constant. This set consists of three equations which are formally known as equations of motion. The equations are derived below.

Consider a particle moving with a constant acceleration 'a'. At the time t = 0, particle is at s_0 with velocity 'u' (initial velocity). The particle acquires velocity 'v' (final velocity) after the time 't'.

FIRST EQUATION OF MOTION: We know that acceleration is the rate of change of velocity,

$$a = \frac{dv}{dt}$$

$\Rightarrow \quad dv = adt$

$\Rightarrow \quad \int_u^v dv = a\int_0^t dt$

$\Rightarrow \quad [v]_u^v = a[t]_0^t$

$\Rightarrow \quad v - u = at$

$\Rightarrow \quad \boxed{v = u + at}$

SECOND EQUATION OF MOTION: We know that velocity is the rate of change of displacement, let the displacement be 's',

$$v = \frac{ds}{dt}$$

$\Rightarrow \quad ds = vdt \qquad$ (i)

But, v = u + at, put this in equation (i)

$\Rightarrow \quad ds = (u + at)dt$

$\Rightarrow \quad \int_{s_0}^{s} ds = \int_0^t (u + at)dt$

$\Rightarrow \quad [s]_{s_0}^{s} = \int_0^t udt + a\int_0^t tdt$

$\Rightarrow \quad [s - s_0] = u[t]_0^t + \frac{a}{2}[t^2]_0^t$

$\Rightarrow \quad \boxed{s - s_0 = ut + \frac{1}{2}at^2}$

THIRD EQUATION OF MOTION: Eliminate 't' using the first and second equation of motion. Put $t = \frac{v - u}{a}$ in second equation of motion. We will arrive at the third equation of motion which is given below.

$\Rightarrow \quad s - s_0 = u\left(\frac{v - u}{a}\right) + \frac{1}{2}a\left(\frac{v - u}{a}\right)^2$

$\Rightarrow \quad 2a(s - s_0) = 2uv - 2u^2 + v^2 + u^2 - 2uv$

$$\boxed{v^2 = u^2 + 2a(s - s_0)}$$

DISPLACEMENT IN n^{th} SECOND

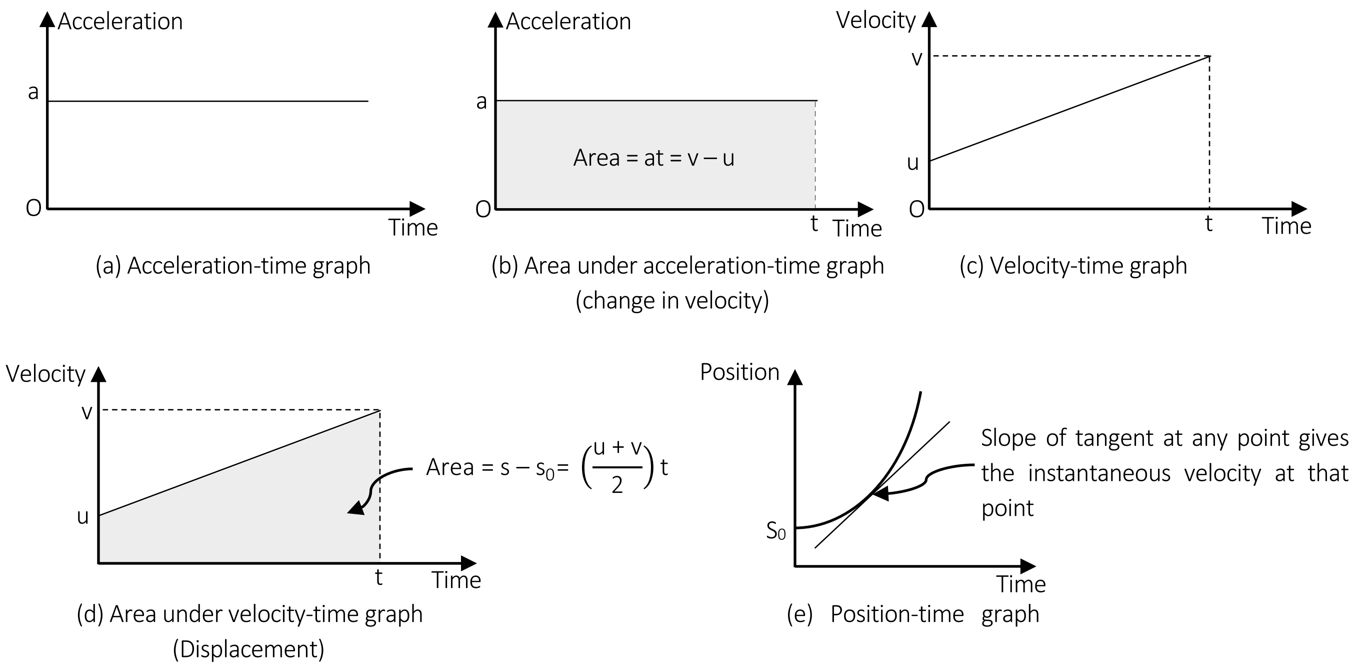
Displacement of particle in n seconds, $\quad s_n = un + \frac{1}{2}an^2$

Displacement of particle in n - 1 seconds, $\quad s_{n-1} = u(n - 1) + \frac{1}{2}a(n - 1)^2$

Displacement in n^{th} second, $\quad s_{n^{th}} = s_n - s_{n-1} = un + \frac{1}{2}an^2 - u(n - 1) - \frac{1}{2}a(n - 1)^2$

$$s_{n^{th}} = u + \frac{a}{2}(2n - 1)$$

Acceleration-time and velocity-time graphs for uniformly accelerated motion are given below. The area between acceleration-time graph and the time axes equals to change in velocity. The area between velocity-time graph and the time axes equals to change in position. Therefore, position–time relation or position at any instant can be obtained. Slope of tangent at any point on position-time graph gives the instantaneous velocity at that point. Slope of tangent of velocity-time graph gives the acceleration of the particle. Displacement–time graph for uniformly accelerated or retarded motion is a parabola. Since, for constant acceleration, then relation between displacement and time (second equation of motion) is quadratic in nature.



(a) Acceleration-time graph

(b) Area under acceleration-time graph (change in velocity)

(c) Velocity-time graph

(d) Area under velocity-time graph (Displacement)

(e) Position-time graph

Figure 15: Acceleration-time, velocity-time and position-time graphs for uniformly accelerated motion

Illustration 10: Consider a particle moving in straight line with constant acceleration "a" traveling 50 m in 5^{th} second and 100 m in 10th second. Find

(a) Initial velocity (u)
(b) Acceleration (a)
(c) Displacement till 7 s
(d) Velocity after 7 s
(e) Displacement between t = 6 s and t = 8 s **[JEE MAIN]**

Sol: For 5th second and 10th second, we get two equations and two variables u and a. So, we solve the equations to get the values of u and a.

Given $s_{5^{th}}$ = 50 m and $s_{10^{th}}$ = 100 m

Now, $s_{5^{10}} = 50 = u + \frac{a}{2}[2(5) - 1]$

$\Rightarrow$ $2u + 9a = 100$ (i)

and, $s_{10^{th}} = 100 = u + \frac{a}{2}[2(10) - 1]$

$\Rightarrow$ $2u + 19a = 200$ (ii)

From (i) and (ii), we get,

$a = 10\ ms^{-2}$ and $u = 5\ ms^{-1}$

Now, $s_7 = (5)(7) + \frac{1}{2}(10)(7)^2 = 35 + 245 = 280$ m

Also, Displacement between 6 and 8 seconds = $s_8 - s_6 = \left(ut + \frac{1}{2}at^2\right)_{t=8} - \left(ut + \frac{1}{2}at^2\right)_{t=6} = u(8-6) + \frac{1}{2}a\left[8^2 - 6^2\right]$

$$= (5)(2) + \left[\frac{1}{2} \times 10 \times 28\right] = 150 \text{ m}$$

Illustration 11: Consider a particle moving in a straight line with constant acceleration, has a velocity $v_P = 7$ ms^{-1} and $v_Q = 17$ ms^{-1}, when it crosses the point P and Q respectively. Find the speed of the particle at mid-point of PQ. **[JEE ADVANCED]**

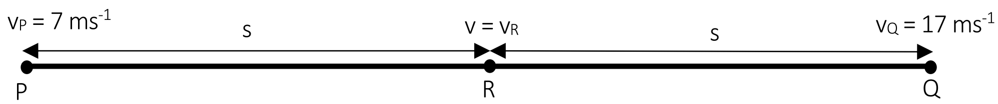


Figure 16

Sol: Let the mid-point be R

Let, PR = RQ = s

Using third equation of motion,

We get, $v_R^2 = 7^2 + 2as$ (i)

and, $17^2 = v_R^2 + 2as$ (ii)

From equations (i) and (ii), we get,

$2v_R^2 = 338$

⇒ $v_R = 13$ ms^{-1}

Illustration 12: A body moving with uniform acceleration covers 24 m in the 4th second and 36 m in the 6th second. Calculate the acceleration and initial velocity. **[JEE MAIN]**

Sol: We know the formula for displacement in nth second. For 4th second and 6th second, we get two equations and two variables u and a. So, we solve the equations to get the values of u and a.

We know, $s_{n^{th}} = u + \frac{a}{2}(2n - 1)$

∴ $24 = u + \frac{a}{2}[(2)(4) - 1]$ (i)

and $36 = u + \frac{a}{2}[(6)(2) - 1]$ (ii)

From equation (i) and (ii) we get, $a = 6$ ms^{-2} and $u = 3$ ms^{-1}

Illustration 13: Acceleration-time graph of a particle moving in a straight line is shown in the figure. At time t = 0, velocity of the particle is 2 ms^{-1}. Find velocity at the end of the 4th second. **[JEE MAIN]**

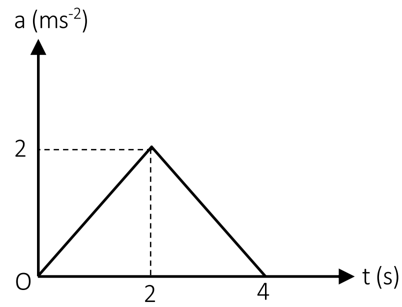


Figure 17: Acceleration-time graph

Sol: The area enclosed by the acceleration-time graph between t = 0 and t = 4s will give the change in velocity in this time interval.

Change in velocity = Area under acceleration-time graph

Hence, $v_f - v_i = \frac{1}{2}(2)(2)(4) = 8\ ms^{-1}$

∴ $v_f = v_i + 8 = (2 + 8)\ ms^{-1} = 10\ ms^{-1}$

NOTE

The following are the lists of motions that are not possible practically:

Slopes of velocity-time or displacement-time graphs can never be infinite at any point, because infinite slope of velocity-time graph means infinite acceleration. Similarly, infinite slope of displacement-time graph means infinite velocity. Hence, the graphs shown here are not possible.

At a particular time, two values of velocities v_1 and v_2 or displacements s_1 and s_2 are not possible. Hence, the following graphs shown here are not possible.

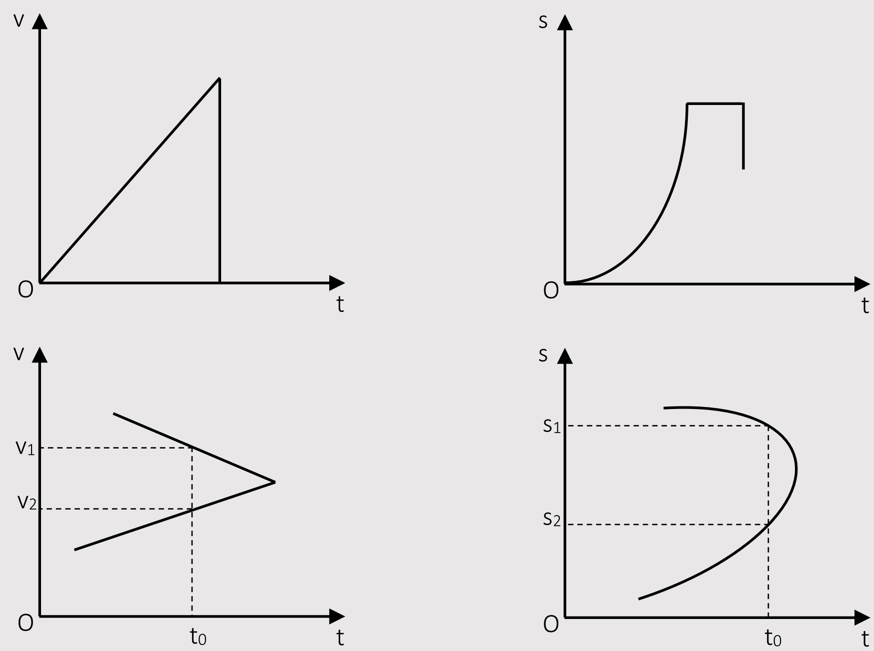


Figure 18

FREE FALL

Due to the phenomena of gravitation, which we will learn later, the earth exerts a force of attraction on all the bodies called the force of gravity because of which some acceleration is induced which is called acceleration due to gravity, represented by g. All bodies irrespective of their size, weight or composition fall with the same acceleration near the surface of earth in the absence of air. Motion of a body falling towards the earth from a small altitude is known as free fall.

6.2.1 MOTION OF A PARTICLE UNDER GRAVITY

The presence of gravity has effect on the motion of the particle. The acceleration due to gravity always acts downwards towards the surface of the earth for any particle undergoing motion in the vertical direction. The value of acceleration due to gravity 'g' is 9.8 ms^{-2}. All the three equations of motion are applicable for the particle under the influence of gravity and the acceleration of the particle is 'g' towards the surface of the earth.

(A) PARTICLE PROJECTED VERTICALLY UPWARDS

Considering the point of projection as origin and direction of motion (vertically up) as positive, the acceleration of the particle is '- g'. If a body is projected with velocity 'u' and after time 't' it reaches to a height 'h' then the equations of motion are as follows.

$$v = u - gt \qquad \text{(i)}$$

$$h = ut - \frac{1}{2}gt^2 \qquad \text{(ii)}$$

$$v^2 = u^2 - 2gh \qquad \text{(iii)}$$

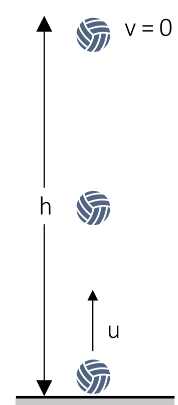


Figure 19

When the particle reaches the maximum height, its velocity becomes zero. The maximum height can be calculated using equation (iii) as follows.

$$0^2 = u^2 - 2gH$$

$$\Rightarrow \quad \boxed{H = \frac{u^2}{2g}} \qquad \text{Maximum Height}$$

Now, using equation (i), we get the time to reach the maximum height as follows.

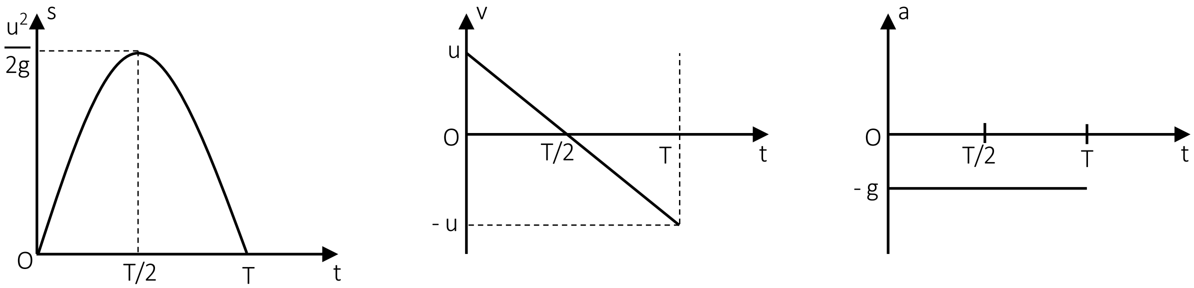
$$0 = u - gt$$

$$\Rightarrow \quad t = \frac{u}{g}$$

The time of flight is the time taken by the object from the beginning of its journey to the point back to its initial coordinate. The time taken by the particle to reach the maximum height is the same as the time taken by the particle to reach the initial point. Hence, the time of flight for the particle is double the time taken by the particle to reach the maximum height.

$$\boxed{T = \frac{2u}{g}} \qquad \text{Time of Flight}$$

The position-time, velocity-time and acceleration-time graphs for the motion of the particle are shown below. The speed with which a body is projected up is equal to the speed with which it comes down. The magnitude of velocity at any point is same whether the body is moving up or down.



(a) Displacement-time graph (b) Velocity-time graph (c) Acceleration-time graph

Figure 20: Displacement-time, velocity-time and acceleration-time graphs for a particle thrown vertically upwards

(B) PARTICLE DROPPED FROM A HEIGHT

Considering the point of projection as origin and direction of motion (vertically down) as positive, the initial velocity of the particle is zero as the particle starts from rest. The acceleration of the particle is 'g'. The equations of motion are written as follows.

$$v = u + gt \qquad \text{(i)}$$

$$h = ut + \frac{1}{2}gt^2 \qquad \text{(ii)}$$

$$v^2 = u^2 + 2gh \qquad \text{(iii)}$$

Suppose the particle was dropped from a height H. The time of flight of the particle is the time taken by the particle to reach the ground. Using equation (ii), the timeof flight can be calculated as follows.

$$H = (0)T + \frac{1}{2}gT^2$$

$\Rightarrow$ $\boxed{T = \sqrt{\frac{2H}{g}}}$ **Time of Flight**

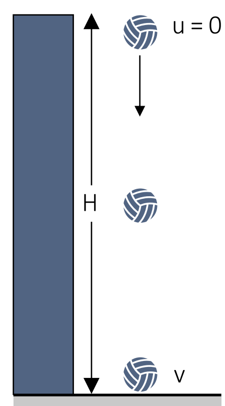


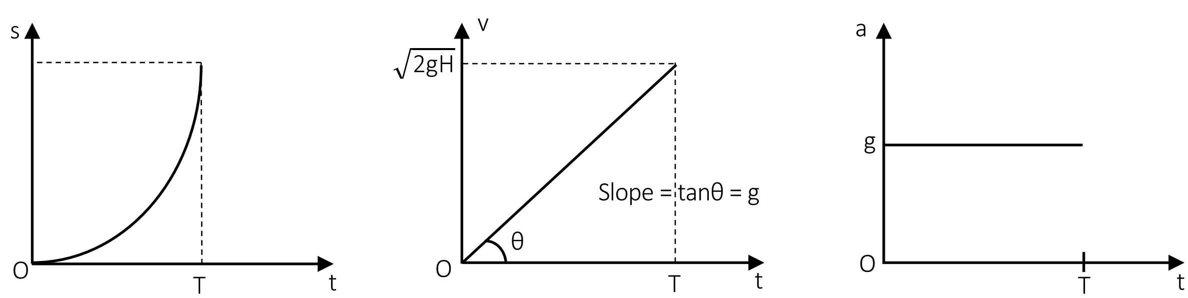
Figure 21

Also, the velocity of the particle at the lowest point on the groung can be calculated using equation (iii) as follows.

$$v^2 = 0^2 + 2gH$$

$\Rightarrow$ $\boxed{v = \sqrt{2gH}}$ **Final Velocity**

The position-time, velocity-time and acceleration-time graphs for the motion of the particle are shown below.



(a) Displacement-time graph (b) Velocity-time graph (c) Acceleration-time graph

Figure 22: Displacement-time, velocity-time and acceleration-time graphs for a particle dropped from a height

(C) PARTICLE THROWN VERTICALLY FROM A HEIGHT

The particle can thrown vertically in two ways.

First, when the particle is thrown in upward direction with some initial velocity, the velocity of the particle is upwards whereas acceleration acts downwards, i.e. they are in opposite directions. Hence initially the object undergoes retardation and rises through a certain height and then it undergoes free fall from that height.

Second, when the particle is thrown from a certain height with some initial velocity, both the velocity and acceleration are in the downward direction, i.e. velocity and acceleration are in the same direction. In this case, the object undergoes acceleration. It hits the ground with a speed greater than the speed if it had gone through free fall.

Equations of motion are used in both the cases. Assuming a sign convention for the direction, the values of displacement, velocity and acceleration can be placed in the equations with appropriate signs to calculate the values of maximum height, time of flight and velcity at various times of the motion.

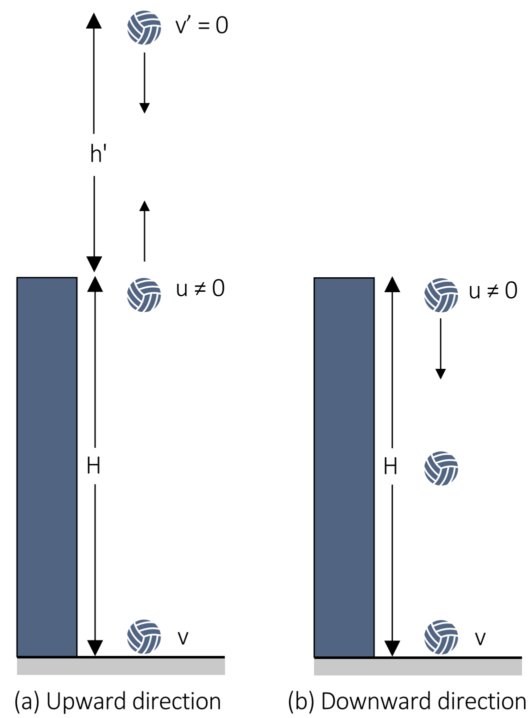


Figure 23: Particle thrown vertically from a height

Illustration 14: Consider a ball being thrown upwards with an initial speed of u. Find out u if the ball is at a height of 80 m and the interval between two times is 6 s. ($g = 10\ ms^{-2}$) **[JEE MAIN]**

Sol: Body thrown vertically upwards reaches the maximum height, stops momentarily and then starts falling vertically downwards. So, for any point at height less than the maximum height, the body will reach the point twice during its travel, first time while ascending and the second time while descending.

$u = u\ ms^{-1}$, $a = -g = -10\ ms^{-2}$ and $s = 80$ m

We know, $$s = ut + \frac{1}{2}at^2$$

∴ $$80 = ut - \frac{10}{2}t^2$$

Or, $$5t^2 - ut + 80 = 0$$

⇒ $$t = \frac{u + \sqrt{u^2 - 1600}}{10} \text{ and } \frac{u - \sqrt{u^2 - 1600}}{10}$$

Given that, $$t = \frac{u + \sqrt{u^2 - 1600}}{10} - \frac{u - \sqrt{u^2 - 1600}}{10} = 6$$

⇒ $$\frac{\sqrt{u^2 - 1600}}{5} = 6$$

⇒ $$\sqrt{u^2 - 1600} = 30$$

⇒ $$u^2 - 1600 = 900$$

⇒ $$u^2 = 2500$$

⇒ $$u = 50\ ms^{-1}$$

Illustration 15: A ball is thrown upwards from 40 m high tower with a velocity of 10 ms^{-1}. Calculate the time when it strikes the ground. ($g = 10\ ms^{-2}$) **[JEE MAIN]**

Sol: In the second equation of motion with constant acceleration, value of all the quantities need to be substituted with proper sign. If the displacement and acceleration are in the opposite direction of initial velocity (taken as positive) then substitute there values with negative sign.

$u = +10\ ms^{-1}$, $a = -10\ ms^{-2}$ and $s = -40$ m

Substituting in $s = ut + \frac{1}{2}at^2$

⇒ $-40 = 10t - 5t^2$

⇒ $5t^2 - 10t - 40 = 0$

⇒ $t^2 - 2t - 8 = 0$

Solving this, we get t = 4 s and - 2 s. Considering the positive value, t = 4 s.

Illustration 16: A ball is dropped from the top of a building. The ball takes 0.5 s to fall past the 3 m length of a window, which is some distance below the top of the building.

(a) How fast was the ball going as it passed the top of the window?
(b) How far is the top of the window from the point at which the ball was dropped?

Assume acceleration g in free fall due to gravity be 10 ms^{-2} downwards. **[JEE ADVANCED]**

Sol: The ball is dropped, so it start falling from the top of the building with zero initial velocity (u = 0). The motion diagram is shown with the given information in the figure. Let the velocity of the ball at the top of the window be 'v' and the time taken to reach the top of the window be 't'. Using the first equation of the constant acceleration motion, we have

$v = 0 + 10t = 10t$ (i)

For the distance travelled through the window, using the second equation of motion with v = 10t

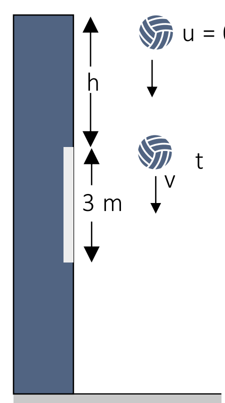


Figure 24

$s = ut' + \frac{1}{2}at'^2$

$t' = 0.5$ s (given)

⇒ $3 = (10t)(0.5) + \frac{1}{2}(10)(0.5)^2$

⇒ $3 = 5t + 1.25$

⇒ $t = 0.35$ s

∴ Using (i), $v = 10 \times 0.35 = 3.5\ ms^{-1}$

Again, using second equation of motion from the starting point to the top of the window

$s = ut + \frac{1}{2}at^2$

$h = \frac{1}{2}(10)(0.35)^2$

⇒ $h = 61.25$ cm

6.3 NON-UNIFORMLY ACCELERATED MOTION

For non-uniformly acceleratted motion, the equation of motion cannot be used. When we derived the equations of motion for uniformaly accelerated motion, we assumed the acceleration to be constant. But for non-uniformly accelerated particles, acceleration is expressed as a function of one or more of the variables t, x and v. Let us consider three common cases.

6.3.1 ACCELERATION AS A FUNCTION OF TIME

If the acceleration of a particle is a given as a function of time, say a = f(t),

We know, $a = \frac{dv}{dt}$

$\Rightarrow$ $dv = f(t)dt$

$\Rightarrow$ $\int dv = \int f(t)dt$ (i)

Equation (i) gives the velocity 'v' as a function of time, say v = g(t),

We know, $v = \frac{dx}{dt}$

$\Rightarrow$ $dx = g(t)dt$

$\Rightarrow$ $\int dx = \int g(t)dt$ (ii)

Equation (ii) gives the displacement as a function of time.

Illustration 17: The acceleration of a particle moving along the x-direction is given by equation a = (3 - 2t) ms^{-2}. At the instants t = 0 and t = 6 s, it occupies the same position.

(a) Find the initial velocity u.
(b) What will be the velocity at t = 2 s? **[JEE MAIN]**

Sol: By substituting the given equation in equation $a = \frac{dv}{dt}$, we have

$$dv = (3 - 2t)dt$$

$\Rightarrow$ $\int_u^v dv = \int_0^t (3 - 2t)dt$

$\Rightarrow$ $v = u + 3t - t^2$ (i)

By substituting the given equation (i) in equation $v = \frac{dx}{dt}$, we have

$$dx = (u + 3t - t^2)dt$$

$\Rightarrow$ $\int_{x_0}^{x} dx = \int_0^t (u + 3t - t^2)dt$

$\Rightarrow$ $x = x_0 + ut + \frac{3}{2}t^2 - \frac{1}{3}t^3$ (ii)

(a) Applying the given condition that the particle occupies the same x coordinate at the instants t = 0 and t = 6 s in equation (ii), we have

$$x_0 = x_6$$

$\Rightarrow \quad x_0 = x_0 + 6u + 54 - 72$

$\Rightarrow \quad u = 3\ ms^{-1}$

(b) Using u in equation (i), we have,

$$v = 3 + 3t - t^2$$

$\Rightarrow \quad v_2 = 5\ ms^{-1}$

6.3.2 ACCELERATION AS A FUNCTION OF POSITION

If acceleration is a given function of position say a = f(x), we have to use equation,

$$a = \frac{dv}{dt} = \frac{dv}{dt}\frac{dx}{dx} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}$$

$\Rightarrow \quad vdv = adx$

Now substituting f(x) for a, we have

$\Rightarrow \quad \int vdv = \int f(x)dx$

The above equation provides us with velocity as function of position. Let relation obtained in this way is v = g(x).

Now substituting g(x) for v in equation $v = \frac{dx}{dt}$, we have

$\Rightarrow \quad dt = \frac{dx}{g(x)}$

$\Rightarrow \quad \int dt = \int \frac{dx}{g(x)}$

The above equation yields the desired relation between x and t.

Illustration 18: Acceleration of a particle moving along the x-axis is defined by the law a = - 4x , where a is in ms^{-2} and x is in meters. At the instant t = 0, the particle passes the origin with a velocity of 2 ms^{-1} moving in the positive x direction.

(a) Find its velocity v as function of its position coordinates.
(b) Find its position x as function of time t.
(c) Find the maximum distance it can go away from the origin. **[JEE MAIN]**

Sol: (a) By substituting given expression in the equation $a = v\frac{dv}{dx}$ and rearranging, we have

$$vdv = -4xdx$$

$\Rightarrow \quad \int_2^v vdv = -4\int_0^x xdx$

$\Rightarrow \quad v = \pm 2\sqrt{1 - x^2} \qquad$ (i)

$\Rightarrow \quad v = 2\sqrt{1 - x^2}$

Since the particle passes the origin with positive velocity of 2 m/s, so the minus sign in the equation (i) has been dropped.

(b) By substituting above obtained expression in the equation $v = \frac{dx}{dt}$ and rearranging, we have

$$\frac{dx}{\sqrt{1-x^2}} = 2t$$

$$\Rightarrow \quad \int_0^x \frac{dx}{\sqrt{1-x^2}} = 2\int_0^t dt$$

$$\Rightarrow \quad \sin^{-1} x = 2t$$

$$\Rightarrow \quad x = \sin 2t$$

(c) The maximum distance it can go away from the origin is 1 m because maximum magnitude of sine function is unity.

6.3.3 ACCELERATION AS A FUNCTION OF VELOCITY

If acceleration is a given function of position say a = f(v), by using the following equation, we can obtain velocity as function of time.

$$a = \frac{dv}{dt}$$

$$\Rightarrow \quad dt = \frac{dv}{f(v)}$$

$$\Rightarrow \quad \int dt = \int \frac{dv}{f(v)}$$

Now using equation $v = \frac{dx}{dt}$, we can obtain position as function of time.

In another way if we use equation $a = v\frac{dv}{dx}$, we can obtain position as function of time.

The above equation provides us with velocity as function of position. Let relation obtained in this way is v = g(x).

$$\Rightarrow \quad dx = \frac{vdv}{f(v)}$$

$$\Rightarrow \quad \int dx = \int \frac{vdv}{f(v)}$$

Now using equation $v = \frac{dx}{dt}$, we can obtain position as function of time.

Illustration 19: Acceleration of particle moving along the x-axis varies according to the law a = - 2v , where a is in ms^{-2} and v is in ms^{-1}. At the instant t = 0, the particle passes the origin with a velocity of 2 ms^{-1} moving in the positive x direction.

(a) Find its velocity v as function of time t.
(b) Find its position x as function of time t.
(c) Find its velocity v as function of its position coordinates.
(d) Find the maximum distance it can go away from the origin.
(e) Will it reach the above-mentioned maximum distance? **[JEE MAIN]**

Sol: (a) By substituting the given relation in equation $a = \frac{dv}{dt}$, we have

$$\frac{dv}{v} = -2dt$$

$\Rightarrow \quad \int_2^v \frac{dv}{v} = -2\int_0^t dt$

$\Rightarrow \quad v = 2e^{-2t}$ (i)

(b) By substituting the above equation in $v = \frac{dx}{dt}$, we have

$dx = 2e^{-2t}dt$

$\Rightarrow \quad \int_0^x dx = 2\int_0^t e^{-2t}dt$

$\Rightarrow \quad x = 1 - e^{-2t}$ (ii)

(c) By substituting given expression in the equation $a = v\frac{dv}{dx}$, and rearranging, we have

$dv = -2dx$

$\Rightarrow \quad \int_2^v dv = -2\int_0^x dx$

$\Rightarrow \quad v = 2(1 - x)$ (iii)

(d) Equation (iii) suggests that it will stop at x = 1 m. Therefore, the maximum distance away from the origin it can go is 1 m.

(e) Equation (ii) suggests that to cover 1 m it will take time whose value tends to infinity. Therefore, it can never cover this distance.

Illustration 20: For a particle moving along x-axis, displacement time equation is $x = 20 + t^3 - 12t$.

(a) Find the position and velocity of the particle at time t = 0
(b) Find out whether the motion is uniformly accelerated or not.
(c) Find out the position of particle when velocity is zero. **[JEE MAIN]**

Sol: Displacement is given as a function of time. Differentiating the equation of displacement w.r.t time we get the velocity as a function of time. Differentiating the equation of velocity w.r.t time we get the acceleration as a function of time.

(a) $x = 20 + t^3 - 12t$ (i)

At t = 0, x = 20 + 0 – 0 = 20 m

By differentiating equation (i) with respect to time i.e.

$v = \frac{dx}{dt} = 3t^2 - 12$ (ii)

Velocity of particle can be obtained at time t.

At t = 0, v = 0 - 12 = - 12 ms^{-1}

(b) Differentiating equation (ii) with respect to time, we get the acceleration,

$a = \frac{dv}{dt} = 6t$

As acceleration is a function of time, the motion is non-uniformly accelerated.

(c) Substituting v = 0 in equation (ii) we get,

$0 = 3t^2 - 12$

From the above equation, t = 2 sec. Substituting it in equation (i) we have,

$$x = 20 + (2)^3 - 12(2)$$

$\Rightarrow$ $x = 4$ m

Illustration 21: Acceleration of an object moving in straight line is $a = v^2$ and initial velocity of that object is u ms^{-1}. Find (i) v(x) i.e. velocity as a function of displacement (ii) v(t) i.e. velocity as a function of time. **[JEE ADVANCED]**

Sol: Acceleration is the differentiation of velocity with respect to time. We can use the following transformation:

$$a = \frac{dv}{dt} = \frac{dv}{dt}\frac{dx}{dx} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}$$

(i) $a = v\frac{dv}{dx} = v^2$

$\Rightarrow$ $\frac{dv}{v} = dx$

$\Rightarrow$ $\int_u^v \frac{dv}{v} = \int_0^x dx$

$\Rightarrow$ $[\ln v]_u^v = [x]_0^x$

$\Rightarrow$ $\ln v - \ln u = x$

$\therefore$ $v = ue^x$

(ii) $a = \frac{dv}{dt} = v^2$

$\Rightarrow$ $\frac{dv}{v^2} = dt$

$\Rightarrow$ $\int_u^v \frac{dv}{v^2} = \int_0^t dt$

$\Rightarrow$ $\left[-\frac{1}{v}\right]_u^v = [t]_0^t$

$\Rightarrow$ $-\left[\frac{1}{v} - \frac{1}{u}\right] = t$

$\therefore$ $v = \frac{u}{1 - ut}$

7. RELATIVE MOTION (Motion in More Than One Dimension Starts Here)

The concept of reference frames was first introduced to discuss relative motion in one or more dimensions. When we say an object has a certain velocity, then this velocity is with respect to some frame that is known as the reference frame. In everyday life, when we measure the velocity of an object, the reference frame is taken to be the ground or the earth. The motion observed by the observer depends on the location (frame) of the observer. This type of motion is called relative motion.

7.1 TYPES OF FRAMES OF REFERENCE

(a) Inertial frames of reference: It is defined as the frame of reference with uniform velocity (both in magnitude and direction). Thus, acceleration is also zero, i.e. $\vec{v}$ = constant and $\vec{a} = 0$.

(b) Non-Inertial frames of reference: It is defined as the frames of reference with non-uniform velocity (either magnitude or direction is not constant). Thus, acceleration is also non-zero, i.e. $\vec{a} \neq 0$.

7.2 RELATIVE VELOCITY

Consider two bodies A and B moving with velocities $\vec{v}_A$ and $\vec{v}_B$ respectively with respect to ground (here ground is an inertial frame of reference, assuming the ground is stationary), then the relative velocity of A with respect to B is given as follows.

$$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$$

Velocity of A with respect to B = Velocity of A – Velocity of B

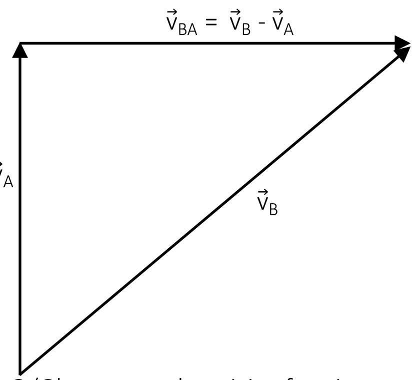


Figure 25: Relative Velocity

Similarly, $\vec{v}_{BA} = \vec{v}_B - \vec{v}_A = -\vec{v}_{AB}$

∴ $\boxed{\vec{v}_{BA} = -\vec{v}_{AB}}$

Relative acceleration: The relative rate of change of $\vec{v}_{BA}$ gives relative acceleration of B with respect to A and is given by following equation.

$$\vec{a}_{AB} = \vec{a}_A - \vec{a}_A$$

Similarly, $\vec{a}_{BA} = \vec{a}_B - \vec{a}_A = -\vec{a}_{AB}$

∴ $\boxed{\vec{a}_{BA} = -\vec{a}_{AB}}$

NOTE

Distance between two objects with is independent of the reference frame. If 'x' is the minimum distance between the two objects at time 't' then in any frame of reference the minimum distance of the objects remains constant at time 't'.

Illustration 22: A man whose velocity in still water is 5 ms^{-1} swims from point A to B (100 m down-stream of A) and back to point A. Velocity of the river is 3 ms^{-1}. Find the time taken in going down-stream and upstream and the average speed of the man during the motion. **[JEE MAIN]**

Sol: The velocity of man in ground frame is the vector sum of the velocity of river and the velocity of man in river frame. In going down-stream, the magnitude of velocity of river and the magnitude of velocity of man in river frame are added to get the magnitude of velocity of man in ground frame. In going up-stream, the magnitude of velocity of river is subtracted from the magnitude of velocity of man in river frame to get the magnitude of velocity of man in ground frame.

The velocity of man in still water = $\vec{v}_m = 5\ ms^{-1}$

Velocity of river = $\vec{v}_r = 3\ ms^{-1}$

Velocity of man during down-stream = $\vec{v}_m + \vec{v}_r = 5 + 3 = 8\ ms^{-1}$

Time taken going down-stream = $\frac{100}{8} = 12.5$ s

Velocity of man during up-stream = $\vec{v}_m - \vec{v}_r = -5 + 3 = -2\ ms^{-1}$

Time taken going up-stream = $\frac{100}{2} = 50$ s

Average Speed = $\frac{100 + 100}{12.5 + 50} = \frac{200}{62.5} = 3.2\ ms^{-1}$

Illustration 23: Yashwant started moving with constant speed 10 ms^{-1} to catch the bus. When he was 40 m away from the bus, it started moving away from him with acceleration of 2 ms^{-2}. Find whether Yashwant catches the bus or not. If yes, at what time he catches the bus. If no, then find the minimum distance between the bus and him. **[JEE MAIN]**

Sol: This problem is best solved in the reference frame of bus. In this frame the initial velocity of Yashwant is towards the bus (assumed positive) and the acceleration of Yashwant is in the opposite direction i.e. away from the bus (assumed negative). Yashwant moves with the initial velocity of 10 ms^{-1} and acceleration of 2 ms^{-2}. Assuming Yashwant never catches the bus, let us find the distance at which his velocity becomes zero.

We know, $v^2 = u^2 + 2as$

Here, $v = 0,\ u = 10\ ms^{-1},\ a = -2\ ms^{-2}$

$\therefore$ $0^2 = (10)^2 + 2(-2)\ s$

$\Rightarrow$ $s = \frac{100}{4} = 25$ m

Our assumption was right, since Yashwant travels only 25 m in the positive direction in bus reference frame. So, minimum distance = 40 – 25 = 15 m.

7.3 APPLICATIONS OF RELATIVE VELOCITY

Relative motion is widely used in two- and three-dimensional motions. The four types of problems arising based on relative motion are as follows.

(a) Problems on minimum distance between two bodies in motion
(b) River–boat problems
(c) Aircraft–wind problems
(d) Rain problems

7.3.1 MINIMUM DISTANCE BETWEEN TWO BODIES IN MOTION

Minimum distance between two moving bodies or the time taken when one body overtakes the other can be solved easily by the principle of relative motion. Here we consider one body to be at rest and other body to be in relative motion of the other body. By combining two problems into one, the solution becomes easy. Following examples will illustrate the statement.

Illustration 24: Car A and car B start moving simultaneously in the same direction along the line joining them. Car A moves with a constant acceleration a = 4 ms^{-2}, while car B moves with a constant velocity v = 1 ms^{-1}. At time t = 0, car A is 10 m behind car B. Find the time when car A overtakes car B. **[JEE MAIN]**

Sol: This problem is best solved in the reference frame of any one of the two cars (say car B). In this frame the initial velocity of car A is in the direction away from car B (assumed negative) and the acceleration of car A is in the direction towards the car B (assumed positive).

Given: $u_A = 0$, $u_B = 1\ ms^{-1}$, $a_A = 4\ ms^{-2}$ and $a_B = 0$

Assuming car B to be at rest, we have

$$u_{AB} = u_A - u_B = 0 - 1 = -1\ ms^{-1}$$

$$a_{AB} = a_A - a_B = 4 - 0 = 4ms^{-2}$$

Now, the problem can be solved easily as follows:

Substituting the proper values in equation,

$$s = ut + \frac{1}{2}at^2$$

We get, $$10 = -t + \frac{1}{2}(4)t^2$$

⇒ $$2t^2 - t - 10 = 0$$

⇒ $$t = \frac{1 \pm \sqrt{1+80}}{4} = \frac{1 \pm \sqrt{81}}{4} = \frac{1 \pm 9}{4}$$

⇒ $$t = 2.5 \text{ or } -2$$

Ignoring the negative value, the desired time is 2.5 s.

7.3.2 RIVER–BOAT PROBLEM

In this problem, we have velocity of river, velocity of boat and velocity or boat with respect to river.

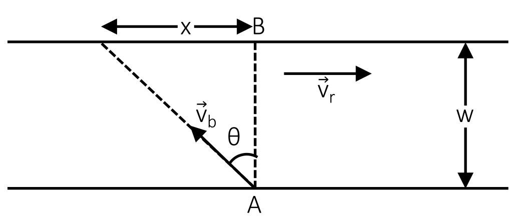


Figure 26: River-Boat Problem

Velocity of river = $\vec{v}_r$

Velocity of boat still water = $\vec{v}_b = 5\ ms^{-1}$

Velocity of boat with respect to river = $\vec{v}_{br} = \vec{v}_b - \vec{v}_r$

Width of the river = w

Angle made by the resultant velocity and the width of the river = θ

Drift made by the boat = x

The problems in this section can be solved by resolving the components of velocity in x- and y-axis and looking at the relative and absolute velocities. There are three special cases.

(i) Condition when the boat crosses the river in shortest interval of time

For the boat to reach the other bank of the river in the shortest interval of time, the velocity of the boat in the perpendicular direction to the flow of the river must be maximum. To achieve this, the boatman must drive the boat in the direction perpendicular to the flow of river. The resultant velocity will make some angle with the river flow.

(ii) Condition when the boat wants to reach point just opposite from where he started

In this case, the resultant velocity of the boat with respect to river should be in the direction such that the drift 'x' is zero. To achieve that, the resultant velocity must be perpendicular to the flow of the river as shown in the figure.

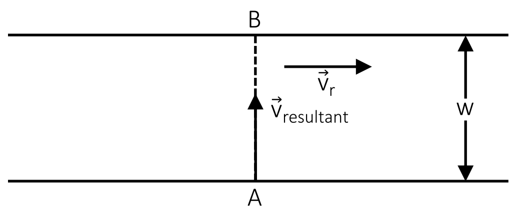


Figure 27: Shortest time for boat

(iii) Shortest path

Distance travelled by the boatman when he reaches the opposite shore is $s = \sqrt{w^2 + x^2}$

Here, w = width of river which is constant. For s to be minimum, modulus of x (drift) should be minimum. Now two cases are possible.

(a) When $v_r < v_b$: In this case x = 0

(b) When $v_r > v_b$: In this case x is minimum

Illustration 25: A man rows a boat at 4 km h^{-1} in still water. If he is crossing a river with a 2 km h^{-1} current.

(a) What will be the direction of the boat, if he wants to reach a point directly opposite the starting point on the other bank?
(b) With these conditions, how much time it will take for him to cross the river, given the width of river is 4 km?
(c) What will be the minimum time and what direction should he head to cross the river in shortest time?
(d) If he wants to row 2 km up the stream and back to the origin, what will be the time required? **[JEE MAIN]**

Sol: The velocity of boat in ground frame is the vector sum of the velocity of river and the velocity of boat in river frame. If the boat heads in the direction perpendicular to the direction of river flow in the frame of the river, it will cross the river in shortest time. But in doing so it will get drifted in the direction of the river flow as well, and thus will not reach the other bank directly opposite to the starting point.

(a) Given, v_b = 4 km h^{-1} and v_r = 2 km h^{-1}

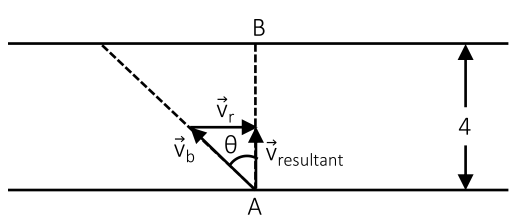


Figure 28: River-Boat Problem

$$\therefore \quad \sin^{-1}\frac{v_r}{v_b} = \sin^{-1}\frac{2}{4} = \sin^{-1}\frac{1}{2} = 30°$$

To reach the point directly opposite to starting point, the boat should head at an angle of 30° with AB or 90° + 30° = 120° with the river flow.

(b) Velocity in the direction perpendicular to the flow = $v_b\cos\theta$

Width of the river = 4 km

Time taken to cross the river = $\frac{4}{v_b\cos\theta} = \frac{4}{4\cos 30°} = \frac{2}{\sqrt{3}}$ h

(c) To cross the river in the shortest time, the boatman should steer the boat in the direction perpendicular to the direction of the flow. Then, the angle θ will be zero.

Shortest time taken to cross the river = $\frac{4}{4}$ = 1 h

(d) Total time = Time upstream + Time downstream

$t = t_1 + t_2$

∴ $t = \frac{2}{4-2} + \frac{2}{4+2} = 1 + \frac{1}{3} = \frac{4}{3}$ h

7.3.3 AIRCRAFT-WIND PROBLEM

This kind of problem is same as the river-boat problem. Let's look at the example to understand it.

Illustration 26: An aircraft flies 400 km h^{-1} in still air. If $200\sqrt{2}$ km h^{-1} wind is blowing from the south and the pilot wants to travel from point A to a point B, north east of A. Find the direction in which the aircraft is to be steered and time of journey if AB = 100 km.

[JEE MAIN]

Sol: The aircraft wants to reach the point B at a distance 100 m from the origin A as shown in the figure. According to the question, the line AB will make an angle 45° with the north direction. Since the wind is blowing towards north, the pilot must steer the aircraft such that the resultant velocity of the aircraft is in the direction of AB. Suppose the aircraft makes an angle α with AB as shown in the figure.

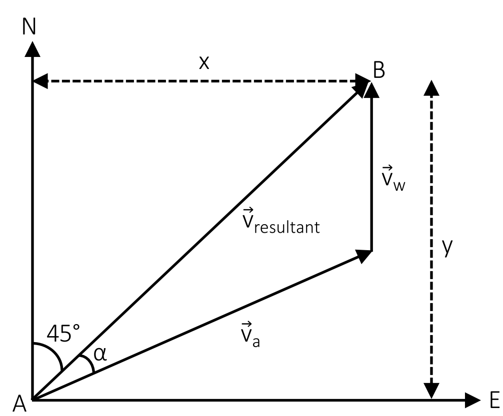


Figure 29

Velocity of aircraft = v_a = 400 km/h

Velocity of wind = $v_w = 200\sqrt{2}$ km/h (north direction)

Distance to be covered in north direction = y = 100cos45°

Distance to be covered in east direction = x = 100sin45°

Resultant velocity of the aircraft in north direction = $v_a \cos(45° + \alpha) + v_w$

Resultant velocity of the aircraft in east direction = $v_a \sin(45° + \alpha)$

Now, using Speed = $\frac{\text{Distance}}{\text{Time}}$

Time taken to cover horizontal distance to reach point B = Time taken to cover vertical distance to reach point B

$$\Rightarrow \quad \frac{100 \sin 45°}{v_a \sin(45° + \alpha)} = \frac{100 \cos 45°}{v_a \cos(45° + \alpha) + v_w}$$

$$\Rightarrow \quad v_a \cos(45° + \alpha) + v_w = v_a \sin(45° + \alpha)$$

$$\Rightarrow \quad 400 \cos(45° + \alpha) + 200\sqrt{2} = 400 \sin(45° + \alpha)$$

$$\Rightarrow \quad 2 \cos(45° + \alpha) + \sqrt{2} = 2 \sin(45° + \alpha)$$

$$\Rightarrow \quad 2[\sin(45° + \alpha) - \cos(45° + \alpha)] = \sqrt{2}$$

$$\Rightarrow \quad [\sin(45° + \alpha) - \sin(45° - \alpha)] = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad 2 \cos 45° \sin \alpha = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \sin \alpha = \frac{1}{2}$$

$$\Rightarrow \quad \alpha = \sin^{-1}\frac{1}{2} = 30°$$

Therefore, the pilot should steer in a direction at an angle of (45° + α) or 75° from north towards east.

Now, time, $$t = \frac{100 \sin 45°}{v_a \sin(45° + \alpha)} = \frac{100\frac{1}{\sqrt{2}}}{400 \sin(75°)} = 1.83 \text{ h}$$

7.3.4 RAIN PROBLEM

This kind of problem is also same as the river-boat problem. Let's look at the example to understand it.

Illustration 27: Rain appears to fall vertically to a man walking at a rate of 3 km h^{-1}. At a speed of 6 km h^{-1}, it appears to meet him at an angle of 45° of vertical. Find out the speed of rain. **[JEE MAIN]**

Sol: This problem is best solved by using Cartesian coordinates. Take x-axis along the horizontal and y-axis vertically upwards. The velocity of man is along positive x-axis. The velocity of rain has both horizontal and vertical components. Express the velocity of man and rain in terms of unit vectors $\hat{i}$ and $\hat{j}$.

Let $\hat{i}$ and $\hat{j}$ be the unit vectors in horizontal and vertical directions, respectively.

Velocity of rain = $\vec{v}_r = x\hat{i} + y\hat{j}$

In the first case, velocity of man = $\vec{v}_m = 3\hat{i}$

$$\therefore \quad \vec{v}_{rm} = \vec{v}_r - \vec{v}_m = (x - 3)\hat{i} + y\hat{j}$$

It seems to be in vertical direction. Hence, x - 3 = 0 or x = 3

In the second case, velocity of man = $\vec{v}_m = 6\hat{i}$

$\therefore \quad \vec{v}_{rm} = \vec{v}_r - \vec{v}_m = (x - 6)\hat{i} + y\hat{j} = -3\hat{i} + y\hat{j}$

This seems to be at 45° of vertical

$\Rightarrow \quad \tan^{-1} 45° = \frac{y}{|-3|}$

$\Rightarrow \quad y = 3$

Therefore, speed of rain is $\vec{v}_r = \sqrt{3^2 + 3^2} = 3\sqrt{2}$ km h^{-1}

8. PROJECTILE MOTION

When an object is projected, making an angle with the horizontal, by an external force in the presence of earth's gravitational field and it continues to move by its own inertia. It is known as projectile and its motion as projectile motion.

For example, a ball hit by a batsman, an arrow shot by an archer, water sprinkling out a water–fountain, an athlete in long jump or high jump, a bullet or an artillery shell fired from a gun.

When any particle is projected to undergo a projectile motion, at least in simple cases such as ball, it does not reach a great height and also does not cover a very large distance, hence the acceleration due to gravity can be assumed to be uniform throughout its motion. Also, the particle projected is very small in size hence the air and wind resistances can also be ignored. Under these conditions, the path of the particle is parabolic. In the figure, a ball thrown to follow a parabolic trajectory is shown as an example of projectile motion.

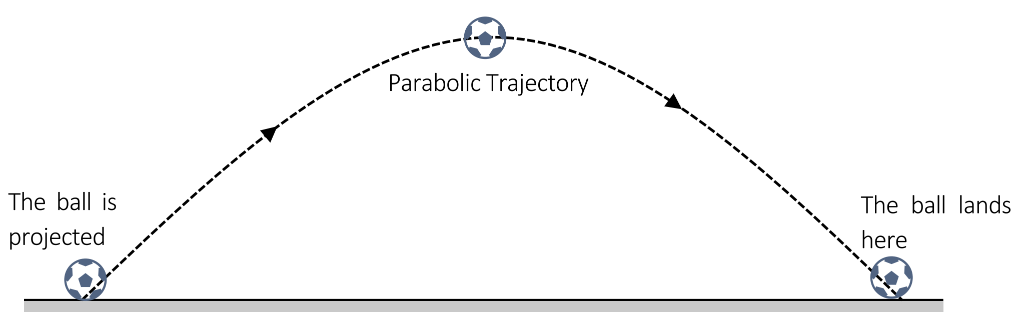


Figure 30: Parabolic trajectory of a projectile

8.1 PROJECTILE MOTION NEAR THE HORIZONTAL OR FLAT GROUND

Consider motion of a ball thrown from ground as shown in the figure. The point from where it is projected is known as point of projection, the point where it falls on the ground is known as point of landing or target. The distance between these two points is known as horizontal range or range, the height from the ground of the highest point it reaches during flight is known as maximum height and the duration for which it remain in the air is known as air time or time of flight. The velocity with which it is thrown is known as velocity of projection and angle which velocity of projection makes with the horizontal is known as angle of projection.

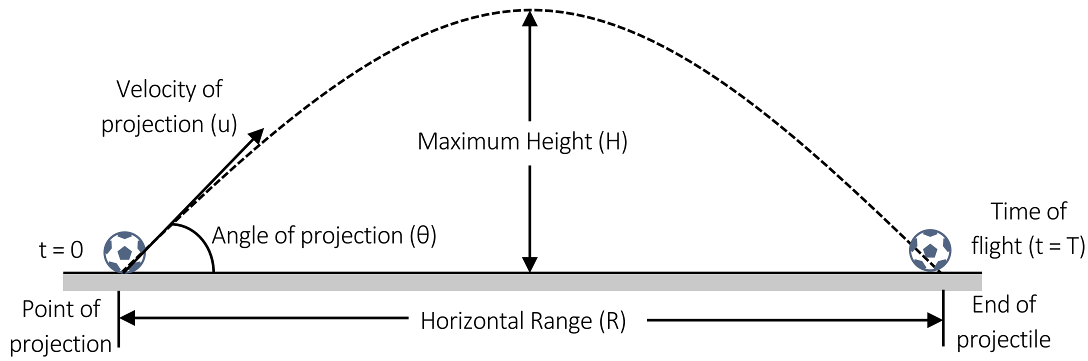


Figure 31: Parabolic trajectory of a projectile

8.1.1 ANALYSING PROJECTILE MOTION

Since parabola is a plane curve, projectile motion on parabolic trajectory becomes an example of a two–dimensional motion. It can be conceived as superposition of two simultaneous rectilinear motions in two mutually perpendicular directions, which can be analysed separately as two Cartesian components of the projectile motion.

When a ball is projected at some angle from the horizontal, its vertical component of velocity decreases in its upward motion and vanishes at the highest point because the acceleration due to gravity is pointing downwards. After reaching the highest point, the vertical component of velocity increases in its downward motion. At the same time, the ball continues to move uniformly in horizontal direction due to inertia i.e., the horizontal component of the velocity does not change since there is no acceleration in the horizontal direction. The actual projectile motion on its parabolic trajectory is superposition of these two simultaneous rectilinear motions. The following figure shows the above explanation in detail.

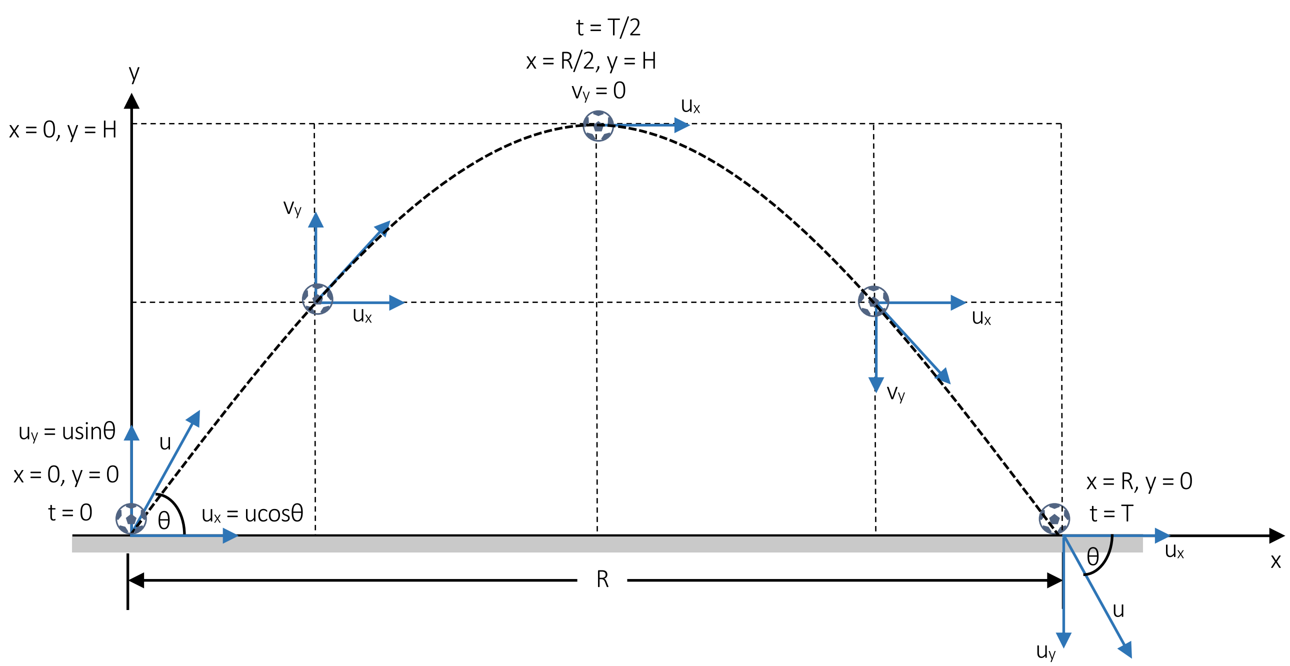


Figure 32: Parabolic trajectory of a projectile (Detailed)

8.1.2 HORIZONTAL COMPONENT OF MOTION

Since the effects of wind and air resistance are assumed negligible as compared to effect of gravity, the horizontal component of acceleration of the ball becomes zero and the ball moves with uniform horizontal component of velocity u_x. This component of motion is described by the following equation.

$$u_x = u \cos \theta$$

$$x = u_x t \qquad \text{(i)}$$

8.1.3 VERTICAL COMPONENT OF MOTION

Component of initial velocity in the vertical direction is u_y. Since forces other than gravitational pull of the earth are negligible, vertical component of acceleration a_y of the ball is g vertically downwards. This component of motion is described by the following three equations. Here v_y denotes y-component of velocity, y denotes position coordinate y at any instant t.

$$v_y = u_y - gt \qquad \text{(ii)}$$

$$y = u_y t - \frac{1}{2}gt^2 \qquad \text{(iii)}$$

$$v_y^2 = u_y^2 - 2gy \qquad \text{(iv)}$$

8.1.4 EQUATION OF TRAJECTORY

Equation of the trajectory is relation between the x and the y coordinates of the ball without involvement of time t. To eliminate t, we substitute its expression from equation (i) into equation (iii).

$$y = x\tan\theta - \frac{g}{2u^2\cos^2\theta}x^2 \qquad \text{(v)}$$

Every projectile motion can be analysed using the above five equations. In a special case of interest, if the projectile lands the ground again, its time of flight, the maximum height reached and horizontal range are obtained using the above equations.

8.1.5 TIME OF FLIGHT

At the highest point of trajectory, the vertical component of the velocity becomes zero. The taken to reach the highest point from ground is same as the time taken to reach the ground again. The time of flight is the total time taken to reach the ground after the projection. Time taken to reach the highest point, say t, can be calculated using equation (ii) as follows.

$$0 = u\sin\theta - gt$$

$$\Rightarrow \qquad t = \frac{u\sin\theta}{g}$$

Time period = 2t

$$\therefore \qquad \boxed{T = \frac{2u\sin\theta}{g}} \qquad \text{Time of Flight}$$

8.1.6 MAXIMUM HEIGHT

At the highest point of trajectory where y = H, the vertical component of velocity becomes zero. Using equation (iv), maximum height of the projectile can be calculated as follows.

$$0^2 = u^2\sin^2\theta - 2gH$$

$$\Rightarrow \qquad \boxed{H = \frac{u^2\sin^2\theta}{2g}} \qquad \text{Maximum Height}$$

8.1.7 HORIZONTAL RANGE

The horizontal range or simply the range of the projectile motion of the ball is distance travelled on the ground in its whole time of flight.

$$R = u_x T = \frac{2u_x u_y}{g}$$

$\Rightarrow$ $\boxed{R = \dfrac{u^2 \sin 2\theta}{g}}$ Horizontal Range

NOTE

For a given value of the initial velocity u, the horizontal range when projected at θ and 90° - θ are equal although times of flight and maximum heights may be different. The proof of the statement is given below.

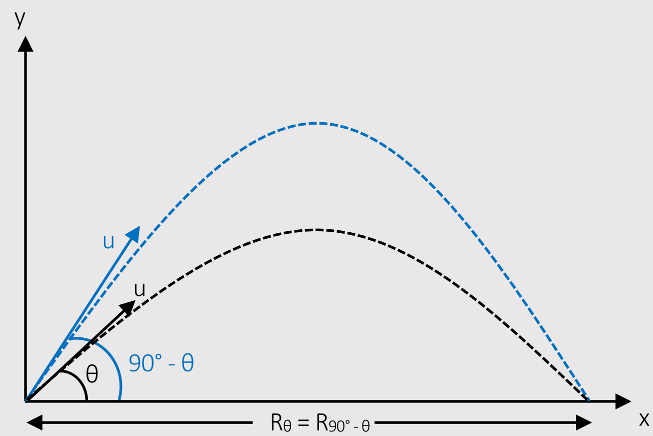


Figure 33: Same horizontal range

$$R_\theta = \frac{u^2 \sin 2\theta}{g}$$

$$R_{90° - \theta} = \frac{u^2 \sin 2(90° - \theta)}{g} = \frac{u^2 \sin 2(180° - 2\theta)}{g} = \frac{u^2 \sin 2\theta}{g} = R_\theta$$

$\Rightarrow$ $\boxed{R_{90° - \theta} = R_\theta}$

Since $a_x = 0$, i.e., motion of the projectile in the horizontal direction is uniform. Hence, horizontal component of velocity u cos θ does not change during its motion. Motion in the vertical direction is first retarded and then accelerated in opposite direction.

8.1.8 MAXIMUM RANGE

It is the maximum distance travelled by a projectile in the horizontal direction for a certain velocity of projection. The above expression of range makes obvious that to obtain maximum range the ball must be projected at angle θ = 45°. Substituting this condition in the expression of range, we obtain the maximum range R_{max}.

$\boxed{R_{max} = \dfrac{u^2}{g}}$ Maximum Horizontal Range

If range is known in advance, the equation of trajectory can be written in an alternative form involving horizontal range as follows.

$$y = x \tan\theta \left(1 - \frac{x}{R}\right)$$

Illustration 28: Assume that a ball is thrown from a field at a speed of 12.0 ms^{-1} and at an angle of 45° with the horizontal. At what distance will it hit the field again? Take g = 10.0 ms^{-2}. **[JEE MAIN]**

Sol: Use the formula for the range of a projectile.

The horizontal range $= \dfrac{u^2 \sin 2\theta}{g} = \dfrac{(12)^2 \times \sin(2 \times 45°)}{10} = \dfrac{144}{10} = 14.4$ m

Thus, the ball hits the field exactly at 14.4 m from the point of projection.

Illustration 29: Find the angle of a projectile for which both the horizontal range and maximum height are equal. **[JEE MAIN]**

Sol: Use the formula for the range and maximum height of a projectile.

Given, R = H

$$\therefore \quad \frac{u^2 \sin 2\theta}{g} = \frac{u^2 \sin^2 \theta}{2g}$$

$$\Rightarrow \quad 2 \sin\theta \cos\theta = \frac{\sin^2\theta}{2}$$

$$\Rightarrow \quad \tan\theta = 4$$

$$\Rightarrow \quad \theta = \tan^{-1} 4$$

Illustration 30: The following figure shows a pirate ship 560 m from a fort defending a harbour entrance. A defence canon, located at sea level, fires balls at initial speed v = 82 ms^{-1}. At what angle θ from the horizontal must a ball be fired to hit the ship? **[JEE ADVANCED]**

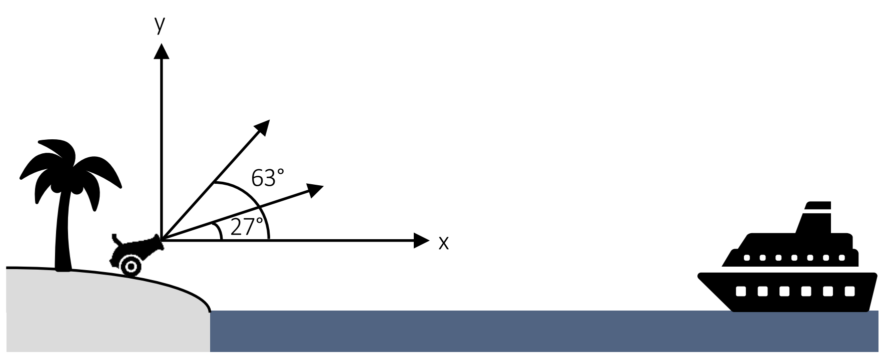


Figure 34

Sol: Use the formula for the range of a projectile to find the angle of projection. We know,

$$R = \frac{u^2 \sin 2\theta}{g}$$

$$\Rightarrow \quad \theta = \frac{1}{2}\sin^{-1}\frac{gR}{v^2} = \frac{1}{2}\sin^{-1}\frac{(9.8)(560)}{(82)^2} = \frac{1}{2}\sin^{-1} 0.816 = \left(\frac{54.7}{2}\right)^{\circ} = 27°$$

The other solution is, 90° - 27° = 63°

Illustration 31: Suppose that a projectile is fired horizontally with a velocity of 98 ms^{-1} from the top of a hill that is 490 m high. Find:

(a) The time taken by the projectile to reach the ground,

(b) The distance of the point where the particle hits the ground from the foot of the hill and

(c) The velocity with which the projectile hits the ground. ($g = 9.8\ ms^{-2}$) **[JEE MAIN]**

Sol: Let x-axis be along the horizontal and the y-axis be along the vertical. The projectile will have uniform velocity along the positive x-axis and uniform acceleration along the negative y-axis. In this problem, we cannot apply the formulae of R, H and T directly. Necessarily we have to follow the three steps discussed in the theory. Here, however, it will be more convenient to choose x and y directions as shown in the figure below.

Here, $u_x = 98\ ms^{-1}, a_x = 0, u_y = 0$ and $a_y = g\ ms^{-2}$

(a) At A, $s_y = 490$ m. Hence applying the convenient equation of motion.

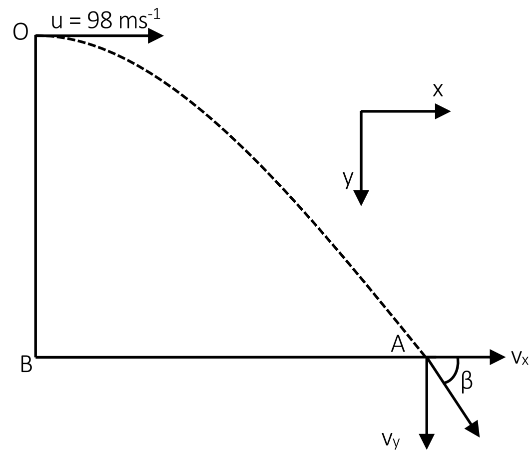


Figure 35

$$s_y = u_y t + \frac{1}{2}a_y t^2$$

$\Rightarrow$ $$490 = 0 + \frac{1}{2}(9.8)t^2$$

$\Rightarrow$ $$t = 10\ s$$

(b) $$BA = s_x = u_x t$$

$\Rightarrow$ $$BA = (98)(10)$$

$\Rightarrow$ $$BA = 980\ m$$

(c) $$v_x = u_x = 98\ ms^{-1} \text{ and } v_y = u_y + a_y t = 0 + (9.8)(10) = 98\ ms^{-1}$$

$\therefore$ $$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(98)^2 + (98)^2} = 98\sqrt{2}\ ms^{-1}$$

$\therefore$ $$\tan\beta = \frac{v_y}{v_x} = \frac{98}{98} = 1$$

$\therefore$ $$\beta = 45°$$

Thus, we show that the projectile hits the ground with a velocity $98\sqrt{2}\ ms^{-1}$ at an angle of $\beta = 45°$ with horizontal.

Illustration 32: A ball is thrown with $25\ ms^{-1}$ at an angle 53° above the horizontal. Find its time of flight, maximum height and range. ($g = 10\ ms^{-2}$) **[JEE MAIN]**

Sol: Velocity of projection, $u = 25\ ms^{-1}$

Angle of projection, $\theta = 53°$

Using equations for time of flight T, maximum height H and range R, we have,

$$T = \frac{2u\sin\theta}{g} = \frac{2 \times 20}{10} = 4\ s$$

$$H = \frac{u^2\sin^2\theta}{2g} = \frac{(20)^2}{2 \times 10} = 20\ m$$

$$R = \frac{u^2\sin 2\theta}{g} = \frac{2 \times 15 \times 20}{10} = 60\ m$$

Illustration 33: A ball 4 s after the instant it was thrown from the ground passes through a point P, and strikes the ground after 5 s from the instant it passes through the point P. Assuming acceleration due to gravity to be $9.8\ ms^{-2}$, find height of the point P above the ground. **[JEE MAIN]**

Sol: Time of flight, T = 9 s

Let initial velocity = u

Horizontal component of initial velocity = usin θ

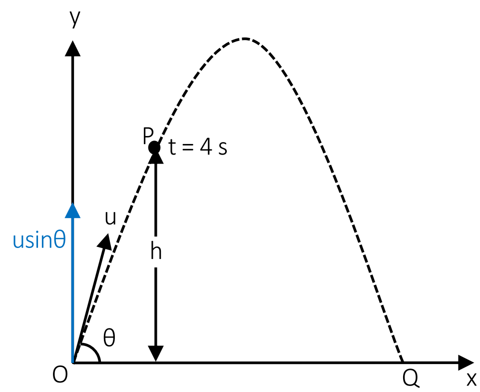


Figure 36

Now, $T = \frac{2u\sin\theta}{g}$

⇒ $9 = \frac{2u\sin\theta}{9.8}$

⇒ $u\sin\theta = \frac{9.8\times 9}{2} = 44.1$ (i)

Suppose the height of the point P is h, and the ball reach there in 4 sec, then applying equation of motion in vertical direction,

$$h = (u\sin\theta)(4) - \frac{1}{2}(9.8)(4)^2 \quad \text{(ii)}$$

Using equations (i) and (ii),

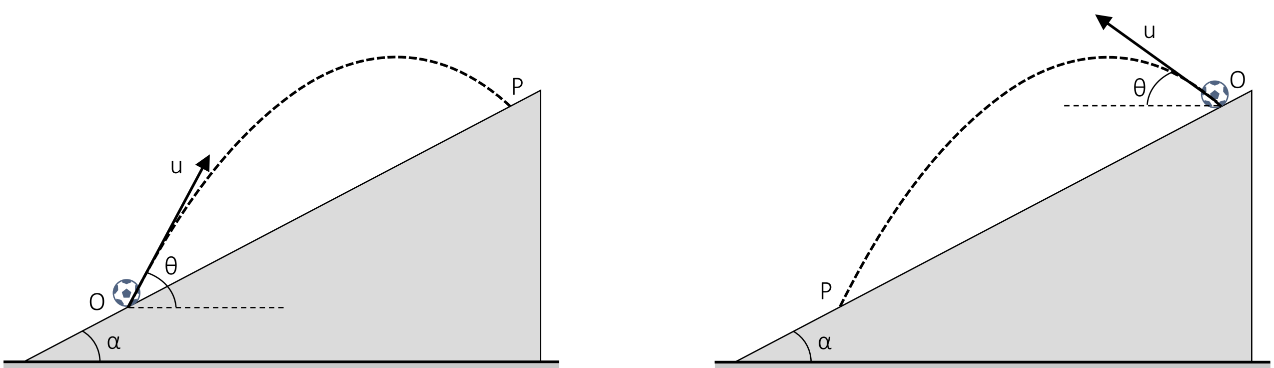
$$h = (44.1)(4) - \frac{1}{2}(9.8)(4)^2$$

⇒ $h = 176.4 - 78.4$

⇒ $h = 98$ m

8.2 PROJECTILE MOTION ON AN INCLINED PLANE

There are various kinds of projectile motions, but an interesting one is projectile on an inclined plane since it has many applications. Artillery application often finds target either up a hill or down a hill. These situations can approximately be modelled as projectile motion up or down an inclined plane.



(a) Projection up the incline

(b) Projection down the incline

Figure 37: Projectile Motion on an Inclined Plane

There are two types of projectiles on an inclined plane as shown in the figure above. Figure (a) shows a particle projected from a point O with velocity u at an angle θ to hit a target at point P up an incline. This projectile motion is called projectile up a hill or inclined plane. Similarly, figure (b) shows a projectile down a hill or inclined plane.

8.2.1 PROJECTILE MOTION UP THE INCLINED PLANE

8.2.1.1 ANALYZING MOTION

There are two approaches of analysing projectile on an inclined plane as follows.

(a) Planar coordinates along the incline and perpendicular to the incline as x- and y- axis.
(b) Planar coordinates in horizontal and vertical directions as x- and y-axis.

We will use the first approach to analyse the motion by choosing the frame of reference accordingly. This makes easier to analyse the motion. We set up our coordinate system along the incline and a direction along the perpendicular to the incline. Consider the projectile motion up an inclined plane. Assume a Cartesian coordinate system whose x-axis coincides with the line of fire OP and the origin with the point of projection as shown in the figure. y-axis is perpendicular to the line OP.

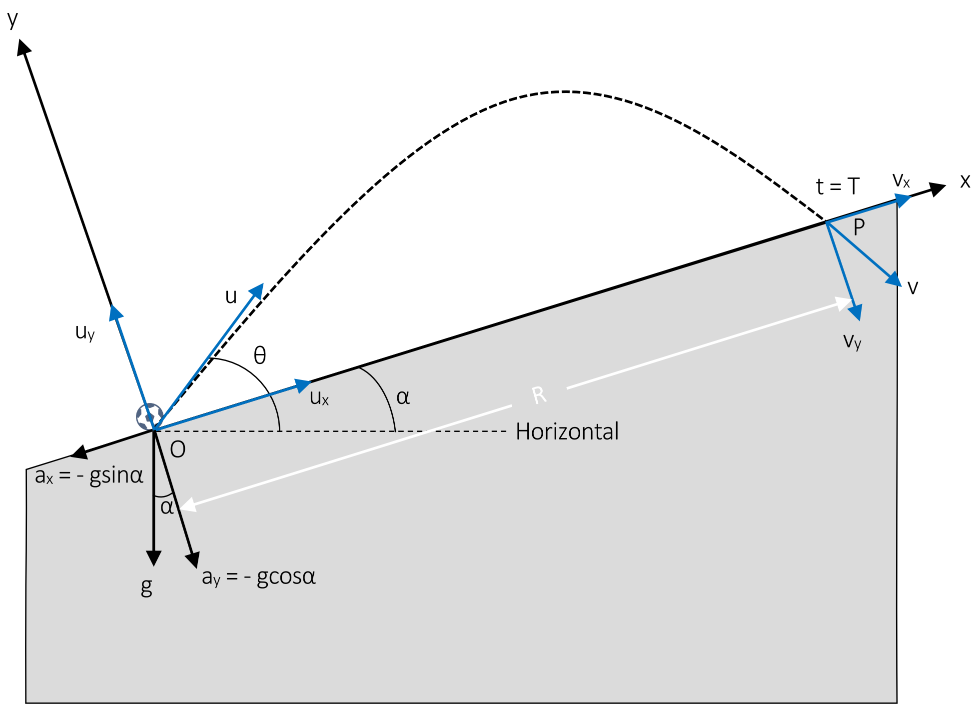


Figure 38: Projectile motion up an inclined plane resolved into its Cartesian components

Velocity of projection makes angle (θ – α) with the positive x-axis, therefore its x and y-components u_x and u_y are as follows.

$$u_x = u\cos(\theta - \alpha)$$

$$u_y = u\sin(\theta - \alpha)$$

At any point on the projectile in the chosen frame of reference, the acceleration due to gravity g being vertical makes the angle α with the negative y-axis, as shown in the figure, therefore x and y-components of acceleration vector are as follows.

$$a_x = -g\sin\alpha$$

$$a_y = -g\cos\alpha$$

8.2.1.2 TIME OF FLIGHT

The time of flight (T) is calculated by analysing motion in y-direction. The displacement in the y-direction after the projectile has returned to the incline is zero. Therefore,

$$y = u_y t + \frac{1}{2}a_y t^2 = 0$$

$$\Rightarrow \quad u\sin(\theta - \alpha)T + \frac{1}{2}(-g\cos\alpha)T^2 = 0$$

$$\Rightarrow \quad T\left[u \sin(\theta - \alpha) + \frac{1}{2}(-g\cos\alpha)T\right] = 0$$

$$\Rightarrow \quad T = 0 \quad \text{and} \quad u\sin(\theta - \alpha) + \frac{1}{2}(-g\cos\alpha)T = 0$$

$$\Rightarrow \quad \boxed{T = \frac{2u\sin(\theta - \alpha)}{g\cos\alpha}} \qquad \text{Time of Flight}$$

8.2.1.3 RANGE OF PROJECTILE

Since, the motion is no more horizontal with respect to ground, we don't call it horizontal range. We can simply say that it is the range of the projectile which is equal to the displacement along the chosen x-axis. Therefore, range can be calculated by applying the following equation.

$$x = u_x t + \frac{1}{2}a_x t^2 = 0$$

Here, x = R (which we will calculate), t = T (we already know). Putting in the values, we get,

$$\Rightarrow \quad R = [u\cos(\theta - \alpha)]\left[\frac{2u\sin(\theta - \alpha)}{g\cos\alpha}\right] + \frac{1}{2}[g\sin\alpha]\left[\frac{2u\sin(\theta - \alpha)}{g\cos\alpha}\right]^2$$

$$\Rightarrow \quad R = \frac{u^2}{g\cos^2\alpha}\{2\cos(\theta - \alpha)\sin(\theta - \alpha)\cos\alpha - 2\sin^2(\theta - \alpha)\sin\alpha\}$$

Using trigonometric relation, $2\sin^2(\theta - \alpha) = 1 - \cos 2(\theta - \alpha)$

$$\Rightarrow \quad R = \frac{u^2}{g\cos^2\alpha}\{\sin[2(\theta - \alpha)]\cos\alpha - \sin\alpha + \cos[2(\theta - \alpha)]\sin\alpha\}$$

Now, we use the trigonometric relation, $\sin(A + B) = \sin A\cos B + \cos A\sin B$,

$$\Rightarrow \quad R = \frac{u^2}{g\cos^2\alpha}\{\sin(2\theta - 2\alpha + \alpha) - \sin\alpha\}$$

$$\Rightarrow \quad \boxed{R = \frac{u^2}{g\cos^2\alpha}\{\sin(2\theta - \alpha) - \sin\alpha\}} \qquad \text{Range of Projectile}$$

8.2.2 PROJECTILE MOTION DOWN THE INCLINED PLANE

For the projectile motion down the incline place, the analysis is very similar to the projectile up the incline plane. The velocities and accelerations along the chosen axes are a little different as shown in the figure and equations given below.

Velocity of projection makes angle (θ – α) with the positive x-axis, therefore its x and y-components u_x and u_y are as follows.

$$u_x = u\cos(\theta + \alpha)$$

$$u_y = u\sin(\theta + \alpha)$$

At any point on the projectile in the chosen frame of reference, the acceleration due to gravity g being vertical makes the angle α with the negative y-axis, as shown in the figure, therefore x and y-components of acceleration vector are as follows.

$$a_x = g\sin\alpha$$

$$a_y = -g\cos\alpha$$

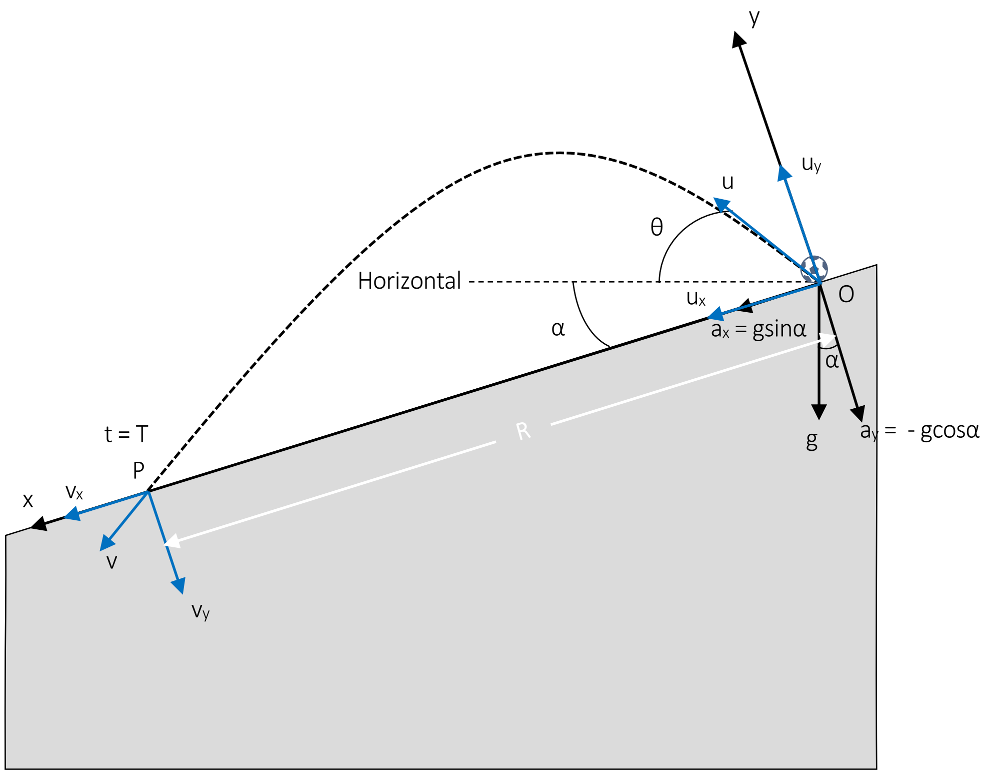


Figure 39: Projectile motion down an inclined plane resolved into its Cartesian components

The expressions for time of flight and range can be derived similar to that of the motion above the inclined plane. The results are given below.

$$T = \frac{2u\sin(\theta + \alpha)}{g\cos\alpha}$$ **Time of Flight**

$$R = \frac{u^2}{g\cos^2\alpha}\{\sin(2\theta + \alpha) + \sin\alpha\}$$ **Range of Projectile**

NOTE

The time of flight and horizontal range expressions for normal projectile can also be obtained by putting the value of α equal to zero in the expression for the projectile motion on an inclined plane as follows.

$$T = \frac{2u\sin(\theta + \alpha)}{g\cos\alpha} = \frac{2u\sin(\theta + 0)}{g\cos 0} = \frac{2u\sin\theta}{g}$$

$$R = \frac{u^2}{g\cos^2\alpha}\{\sin(2\theta + \alpha) + \sin\alpha\} = \frac{u^2}{g\cos^2 0}\{\sin(2\theta + 0) + \sin 0\} = \frac{u^2\sin 2\theta}{g}$$

Illustration 34: Assume that a projectile is thrown from the base of an incline of angle 30° as shown in the figure. It is thrown at an angle of 60° from the horizontal direction at a speed of 10 ms^{-1}. Calculate the total time of flight. (Take g = 10 ms^{-2}). **[JEE MAIN]**

Sol: The x-axis has to be assumed along the inclined. This problem can be handled with a reoriented coordinate system as shown in the figure. Here, the angle of projection with respect to x-direction is (θ – α) and acceleration in y- direction is "g cosα". Now, the total time of flight for the projectile motion, when the point of projection and return are on the same level, is

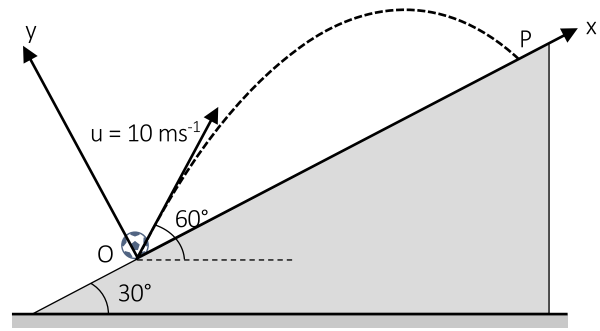


Figure 40

$$T = \frac{2u\sin(\theta - \alpha)}{g\cos\alpha}$$

Now, θ = 60° , α = 30° , and u = 10 ms^{-1}. Then, by substituting these values, we get,

$$\Rightarrow \quad T = \frac{2(10)\sin(60° - 30°)}{g\cos 30°} = \frac{20\sin 30°}{10\cos 30°} = \frac{2}{\sqrt{3}}\ s$$

Illustration 35: Consider that two projectiles are thrown with the same speed from point O and A as shown in the figure so that they hit the incline. If t_O and t_A be the time of flight in two cases, then prove which option out of those given here is true. **[JEE MAIN]**

(A) $t_O = t_A$ (B) $t_O < t_A$ (C) $t_O > t_A$ (D) $t_O = t_A = \frac{u\tan\theta}{g}$

Sol: For the particle thrown from the point O up an incline, the time of flight t_O is,

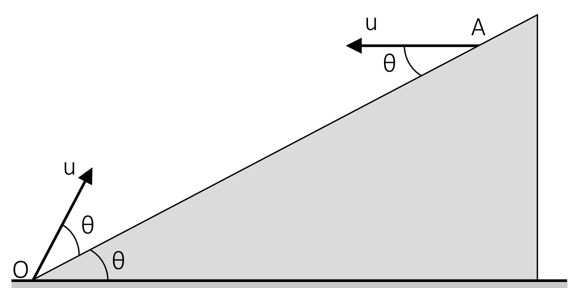


Figure 41

$$t_O = \frac{2u\sin(2\theta - \theta)}{g\cos\theta} = \frac{2u\tan\theta}{g}$$

For the particle thrown from the point A down an incline, the time of flight t_A is,

$$t_A = \frac{2u\sin[2(0) + \theta]}{g\cos\theta} = \frac{2u\tan\theta}{g}$$

$$\Rightarrow \quad t_A = t_O$$

Therefore, option (A) is correct.

Illustration 36: Two inclined planes of angles 30° and 60° are placed so that they touch each other at the base as shown in the figure. Further, a projectile is projected at right angle at a speed of $10\sqrt{3}$ ms^{-1} from point P and hits the other incline at point Q normally. Then, the time of flight is?(Take g = 10 ms^{-2}) **[JEE ADVANCED]**

(A) 1 s (B) 2 s (C) 3 s (D) 4s

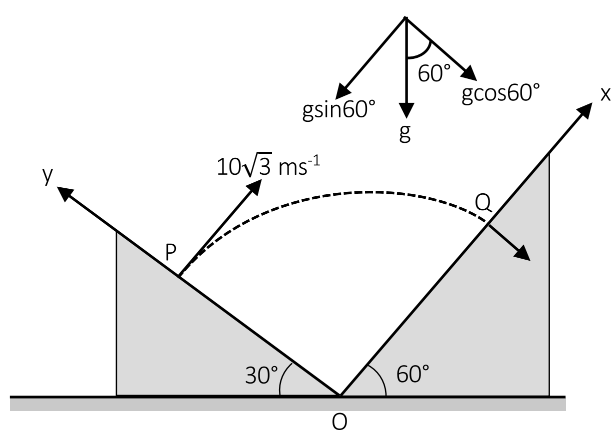


Figure 42

Sol: This problem is a specific case in which the inclined planes are right angles with respect to each other. Therefore, we actually take advantage of this fact in assigning our coordinates along the planes, say y-axis along first incline and x-axis against second incline. Thus, in order to find the time of flight, we can further use the fact that projectile hits the other plane at right angle, i.e., parallel to the y-axis. Therefore, the final velocity along x-axis is zero. Hence, using the equation of motion along axis,

$$v_x = u_x + a_x t \qquad \text{(i)}$$

Here, $v_x = 0$, $u_x = 10\sqrt{3}$, $a_x = -g\sin 60°$ and t is the time of flight T. Put these values in equation (i), we get,

$$0 = 10\sqrt{3} - g \sin 60° \, T$$

$$\Rightarrow \qquad T = \frac{10\sqrt{3}}{g \sin 60°} = \frac{10\sqrt{3}}{10} \times \frac{2}{\sqrt{3}} = 2 \text{ s}$$

NOTE

The maximum range for up and down an incline is when the value of sin(2θ + α) and sin(2θ - α) is one. The ranges for up an incline and down an incline are given below.

$$\left(R_{Up}\right)_{max} = \frac{u^2(1 + \sin\alpha)}{g\cos^2\alpha}$$

$$\left(R_{Down}\right)_{max} = \frac{u^2(1 - \sin\alpha)}{g\cos^2\alpha}$$

9. DESCRIBING MOTION BY ANGULAR VARIABLES

The position of a particle can be completely specified by its position vector $\vec{r}$. If r is the magnitude of the position vector and its orientation relative to some fixed reference direction is θ, then the position of the particle can be shown as in the figure below. A particle P at location shown by position vector $\vec{r} = \overrightarrow{OP}$. Magnitude of the position vector is distance r = OP of the particle from the origin O and orientation of the position vector is the angle θ made by line OP with the positive x-axis. We now specify position of a particle by these to variables r and θ, known as polar coordinates.

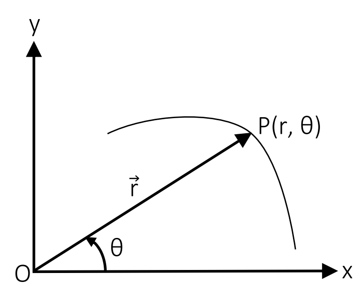


Figure 43: Polar Coordinates

When the particle moves, either or both of these coordinates change with time. If a particle moves radially away from the origin, magnitude r of its position vector $\vec{r}$ increases without any change in angle θ. Similarly, if a particle moves radially towards the origin, r decreases without any change in angle θ. If a particle moves on a circular path with center at the origin, only the angle θ changes with time. If the particle moves on any path other than a radial straight line or circle centered at the origin, both of the coordinates r and θ change with time.

9.1 ANGULAR MOTION

Change in direction of position vector $\vec{r}$ is known as angular motion. It happens when a particle moves on a curvilinear path or straight-line path not containing the origin as shown in the following figures.

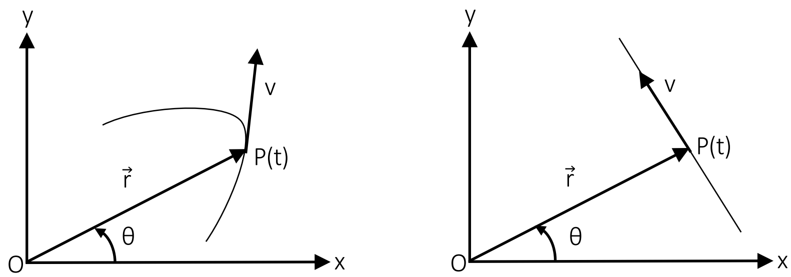


Figure 44: Angular Motion

9.2 ANGULAR POSITION

The coordinate angle θ at an instant is known as angular position of the particle.

9.3 ANGULAR DISPLACEMENT

Angular displacement of a body is the angle in radians (degrees, revolutions) through which a point or line has been turned in a specified sense about a specified axis. Angular displacement is denoted by θ .

9.4 ANGULAR VELOCITY

The instantaneous angular velocity is defined as the rate of change of angular displacement with respect to time. The SI unit of angular velocity is radians per second (rad s^{-1}). Angular velocity is represented by the symbol ω (omega).

$$\boxed{\omega = \frac{d\theta}{dt}}$$ Angular Velocity

9.5 ANGULAR ACCELERATION

The instantaneous rate of change in angular velocity ω with respect to time is known as angular acceleration. In SI units, it is measured in radians per second square (rad s^{-2}), and is usually denoted by the Greek letter α (alpha).

$$\boxed{\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \omega\frac{d\omega}{d\theta}}$$ Angular Acceleration

If a particle moves in a plane, the position vector turns either in clockwise or anticlockwise sense. Assuming one of these directions positive and other negative, problems of angular motion involving angular position θ, angular velocity ω and angular acceleration α can be solved in fashion similar to problems of rectilinear motion involving position x, velocity v, and acceleration a.

10. CIRCULAR MOTION

A particle in circular motion always moves on a circular path. We do not consider rotation in this section which will be covered later. The figure shows a particle P on a circular path of radius r. For simplicity centre of the circular path is assumed at the origin of a coordinate system. Position vector of the particle is shown by a directed radius $\overrightarrow{OP} = \vec{r}$. Hence, it is also known as radius vector. The radius vector is always normal to the path and has constant magnitude and as the particle moves, it is the angular position θ, which varies with time.

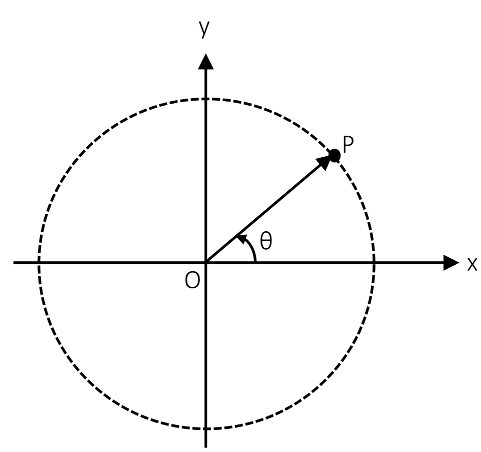


Figure 45: Circular Motion

(A) Angular Variables in Circular Motion: Angular position θ, angular velocity ω and angular acceleration α known as angular variables vary in different manner depending on how the particle moves.

(B) Motion with Uniform Angular Velocity: If a particle moves with constant angular velocity ω, its angular acceleration is zero and position vector turns at constant rate. It is analogous to uniform velocity motion on straight line. The angular position θ at any instant of time t is expressed by the following equation.

$$\boxed{\theta = \theta_0 + \omega t}$$

(C) Motion with uniform angular acceleration: If a particle moves with constant angular acceleration, its angular velocity changes with time at a constant rate. The angular position θ, angular velocity ω and the angular acceleration α bear relations described by the following equations, which have forms similar to corresponding equations that describe uniform acceleration motion.

$$\boxed{\omega = \omega_0 + \alpha t}$$

$$\boxed{\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2}$$

$$\boxed{\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)}$$

(D) Motion with variable angular acceleration: Variable angular acceleration of a particle is generally specified as function of time, angular position or angular velocity. Problems involving variable angular acceleration can also be solved in a way analogous to corresponding rectilinear motion problems in which acceleration is specified as function of time, position or velocity.

10.1 UNIFORM CIRCULAR MOTION

For the motion to be uniform circular motion, the direction of velocity should change continuously without change in magnitude such that the particle always follows a circular path. A change in the direction of velocity implies a change in velocity. Hence, for the particle in uniform circular motion must have some acceleration and hence force. Thus, uniform circular motion signifies presence of force.

In other words, uniform circular motion requires a force, which is always perpendicular to the direction of velocity. Since the direction of velocity is continuously changing, the direction of force, being perpendicular to velocity, should also change continuously. The direction of velocity along the circular trajectory is always tangential in nature. The perpendicular direction to the circular trajectory is, therefore, known as the radial direction. It implies that force (and hence acceleration) in uniform circular motion is radial. For this reason, acceleration in uniform circular motion requires centre, i.e., centripetal (seeking centre). There is no tangential acceleration. The following figure shows the particle moving in uniform circular motion.

10.1.1 LINEAR AND ANGULAR VELOCITY

The instantaneous velocity $\vec{v}$ is also known as linear velocity. In the figure is shown a particle moving on a circular path. As it moves it covers a distance s (arc length).

$$s = \theta r$$

Linear velocity $\vec{v}$ is always along the path. Its magnitude known as linear speed is obtained by differentiating s with respect to time t.

$$v = \frac{d\theta}{dt} r$$

But, $$\frac{d\theta}{dt} = \omega$$

$\Rightarrow$ $\boxed{v = \omega r}$ Relation between Linear and Angular Velocity

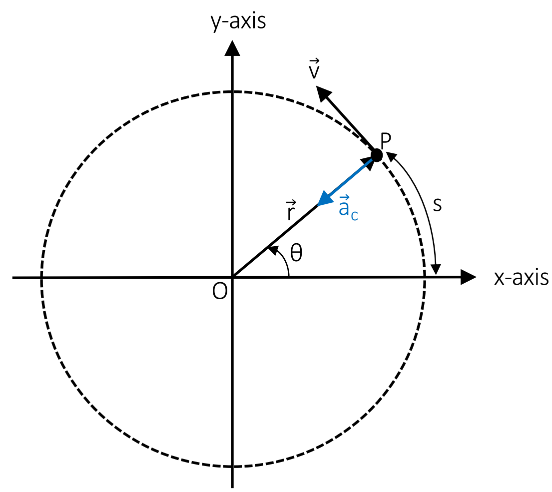


Figure 46: Uniform Circular Motion

10.1.2 CENTRIPETAL ACCELERATION

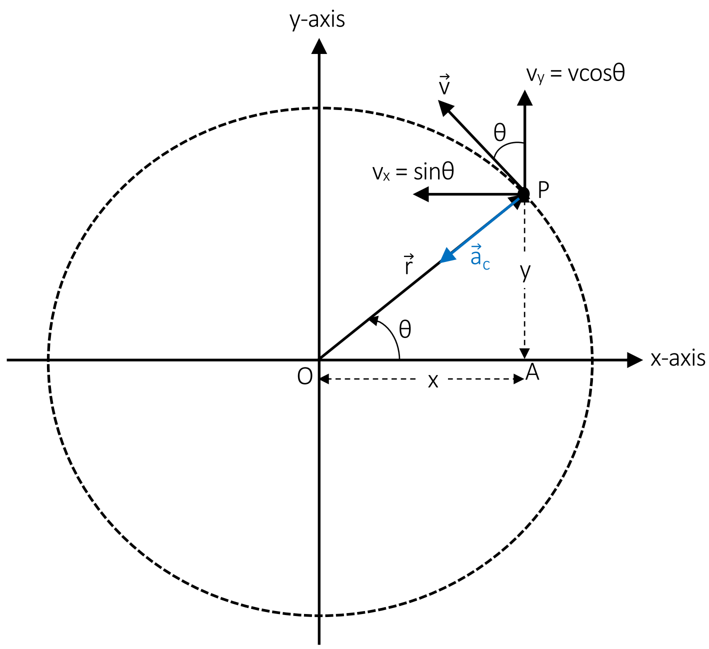


Figure 47: Centripetal acceleration

As shown in the figure above, the coordinates of the particle in cartesian are given by:

$$x = r\cos\theta \text{ and } y = r\sin\theta,$$

Here θ is measured anticlockwise.

The position vector of the position of the particle, r, is represented in terms of unit vectors as:

$$\vec{r} = x\hat{i} + y\hat{j} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} = r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

The magnitude of velocity of the particle is always constant in uniform circular motion. So, the components of velocity, as shown in the figure, are,

$$\vec{v} = v_x\hat{i} + v_y\hat{j} = -v\sin\theta\,\hat{i} + v\cos\theta\,\hat{j}$$

But, $\sin\theta = \frac{y}{r}$ and $\cos\theta = \frac{x}{r}$, therefore,

$$\vec{v} = -\frac{vy}{r}\hat{i} + \frac{vx}{r}\hat{j} = -\frac{v}{r}(y\hat{i} - x\hat{j})$$

Since magnitude of velocity "v" and radius of circle, "r" are constants, we easily differentiate the expression of velocity with respect to time to obtain expression for centripetal acceleration as:

$$\vec{a}_c = -\frac{v}{r}\left(\frac{dy}{dt}\hat{i} - \frac{dx}{dt}\hat{j}\right)$$

$$\Rightarrow \quad \vec{a}_c = -\frac{v}{r}\left(v_y\hat{i} - v_x\hat{j}\right)$$

Substituting the values of component of velocities, we get,

$$\Rightarrow \quad \vec{a}_c = -\frac{v}{r}[(v\cos\theta)\hat{i} - (-v\sin\theta)\hat{j}]$$

$$\Rightarrow \qquad \vec{a}_c = -\frac{v^2}{r}\cos\theta\,\hat{i} - \frac{v^2}{r}\sin\theta\,\hat{j}$$

Therefore, the magnitude of centripetal acceleration is given as follows.

$$a_c = \sqrt{\left(-\frac{v^2}{r}\cos\theta\right)^2 + \left(-\frac{v^2}{r}\sin\theta\right)^2} = \frac{v^2}{r}\sqrt{\cos^2\theta + \sin^2\theta}$$

$$\Rightarrow \qquad \boxed{a_c = \frac{v^2}{r} = \omega^2 r} \qquad \text{Centripetal Acceleration}$$

Centripetal acceleration is also known as radial acceleration.

Illustration 37: Assume that a cyclist negotiates the curvature of 20 m at a speed of 20 ms^{-1}. What is the magnitude of his acceleration? **[JEE MAIN]**

Sol: The speed of the cyclist moving along circular path is constant. So, its acceleration is centripetal.

Let the speed of the cyclist be constant. Then, the acceleration of the cyclist is the centripetal acceleration that is required to move the cyclist along a circular path, i.e., the acceleration resulting from the change in the direction of motion along the circular path.

Hence, v = 20 m/s and r = 20 m

$$\Rightarrow \qquad a_c = \frac{v^2}{r} = \frac{(20)^2}{20} = 20\ ms^{-2}$$

10.2 NON-UNIFORM CIRCULAR MOTION

When the particle is moving in a circle but the magnitude and direction of its velocity both changes with time then it is said to be in non-uniform circular motion. For the particle to follow the circular path, the centripetal acceleration is required. The centripetal acceleration only changes the direction of velocity. To change the magnitude of the velocity, another acceleration called tangential acceleration is required. The following figure shows a particle moving in a non-uniform circular motion.

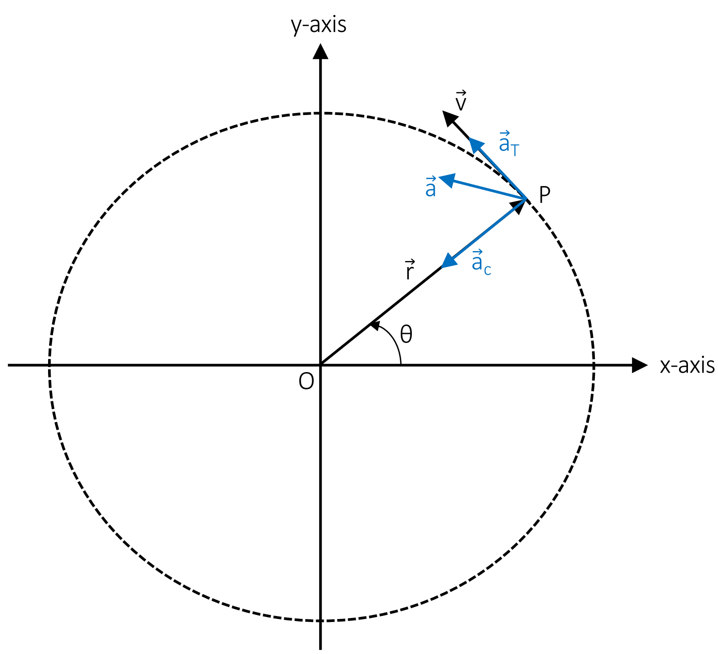


Figure 48: Non-uniform circular motion

10.2.1 TANGENTIAL ACCELERATION

Tangential acceleration is the linear acceleration that helps the particle to gain linear velocity with time. This acceleration is always tangential to the circular path. Mathematically, it is the rate of change of linear velocity of the particle moving in a non-uniform circular motion.

$$\boxed{a_T = \frac{dv}{dt}}$$ Tangential Acceleration

⇒ $$a_T = \frac{d\omega}{dt} r$$

⇒ $$\boxed{a_T = r\alpha}$$ Relation between Linear (Tangential) and Angular Acceleration

The total acceleration of the particle is,

$$\boxed{a = \sqrt{(a_c)^2 + (a_T)^2}}$$ Total Acceleration

Illustration 38: A particle, starting from the position (5 m, 0 m), is moving along a circular path about the origin in x–y plane. The angular position of the particle is a function of time as given here, $\theta = t^2 + 0.2t + 1$. Find the tangential acceleration. **[JEE MAIN]**

Sol: Differentiate the expression for angular position with respect to time to get angular velocity. Tangential acceleration is the product of angular acceleration and the radius. From the data on initial position of the particle, it is clear that the radius of the circle is 5 m. To determine tangential acceleration, we need to have expression of linear speed in time.

Now, $$\omega = \frac{d\theta}{dt} = 2t + 0.2$$

∴ $$v = r\omega = 5 \times (2t + 0.2) = 10t + 1$$

We obtain tangential acceleration by differentiating the above function:

$$a_T = \frac{dv}{dt} = 10\ ms^{-2}$$

Illustration 39: Angular position θ of a particle moving on a curvilinear path varies according to the equation $\theta = t^3 - 3t^2 + 4t - 2$, where θ is in radians and time t is in seconds. What is its average angular acceleration in the time interval t = 2 s to t = 4 s? **[JEE MAIN]**

Sol: Like average linear acceleration, the average angular acceleration α_{avg} equals to ratio of change in angular velocity ω to the concerned time interval Δt.

$$\alpha_{avg} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_i}{t_f - t_i} \qquad \text{(i)}$$

The angular velocity ω being rate of change in angular position can be obtained by equation

$$\omega = \frac{d\theta}{dt} = \frac{d(t^3 - 3t^2 + 4t - 2)}{dt} = 3t^2 - 6t + 4 \qquad \text{(ii)}$$

From the above equation (ii), angular velocities ω_2 and ω_4 at the given instants t = 2 s and 4 s are,

$$\omega_2 = 4\ rad\ s^{-1} \text{ and } \omega_4 = 28\ rad\ s^{-1}$$

Substituting the above values in equation (i), we have $\alpha_{avg} = 12\ rad\ s^{-2}$

Illustration 40: At a particular instant, a particle is moving at a speed of 10 ms^{-1} on a circular path of radius 100 m. Its speed is increasing at the rate of 1 ms^{-2}. What is the acceleration of the particle? **[JEE MAIN]**

Sol: The acceleration of the particle is the vector sum of the centripetal acceleration and the tangential acceleration. The tangential acceleration is equal to the rate of change of speed. The acceleration of a particle is the vector sum of mutually perpendicular centripetal and tangential accelerations. The magnitude of tangential acceleration given here is 1 ms^{-2}. Now, the centripetal acceleration at the particular instant is:

$$a_c = \frac{v^2}{r} = \frac{(10)^2}{100} = 1\ ms^{-2}$$

Hence, the magnitude of the acceleration of the particle is:

$$a = \sqrt{(a_c)^2 + (a_T)^2} = \sqrt{(1)^2 + (1)^2} = \sqrt{2}\ ms^{-2}$$

Illustration 41: A particle starts form rest and moves on a curve with constant angular acceleration of 3.0 rad s^{-2}. An observer starts his stopwatch at a certain instant and record that the particle covers an angular span of 120 rad at the end of 4^{th} second. How long the particle had moved when the observer started his stopwatch? **[JEE MAIN]**

Sol: Let the instants when the particle starts moving and the observer starts his stopwatch, are $t_0 = 0$ to $t = t_1$. Denoting angular positions and angular velocity at the instant $t = t_1$ by θ_1 and ω_1 and the angular position at the instant $t_2 = t_1 + 4s$ by θ_2, we can express the angular span covered during the interval from the following equation.

$$\theta_2 - \theta_1 = \omega_1(t_2 - t_1) + \frac{1}{2}\alpha(t_2 - t_1)^2$$

Substituting values θ_1, θ_2, t_1 and t_2, we have $\omega_1 = 24$ rad s^{-1}

From equation $\omega_1 = \omega_0 + \alpha t_1$, by putting in the value, we have,

$$t_1 = 8\ s$$

Illustration 42: Which of the following expressions represent the magnitude of centripetal acceleration? **[JEE MAIN]**

(A) $\left|\frac{d^2r}{dt^2}\right|$ (B) $\left|\frac{dv}{dt}\right|$ (C) $r\frac{d\theta}{dt}$ (D) None of these

Sol: The magnitude of centripetal acceleration depends on the square of the magnitude of velocity. The expression $\left|\frac{dv}{dt}\right|$ represents the magnitude of tangential acceleration. The differential $\frac{d\theta}{dt}$ represents the magnitude of angular velocity. The expression $r\frac{d\theta}{dt}$ represents the magnitude of tangential velocity and the expression $\frac{d^2r}{dt^2}$ is second-order differentiation of position vector (r). This is the actual expression of acceleration of a particle under motion. Hence, the expression $\left|\frac{d^2r}{dt^2}\right|$ represents the magnitude of total or resultant acceleration. Hence, option (d) alone is correct.

Illustration 43: A particle is executing circular motion. But the magnitude of velocity of the particle changes from zero to $(0.3\hat{i} + 0.4\hat{j})$ ms^{-1} in a period of 1 second. The magnitude of average tangential acceleration is: **[JEE MAIN]**

(A) 0.1 ms^{-2} (B) 0.2 ms^{-2} (C) 0.3 ms^{-2} (D) 0.5 ms^{-2}

Sol: Tangential acceleration is equal to the rate of change of speed. Average tangential acceleration is change in speed divided by total time. The magnitude of average tangential acceleration is the ratio of change in speed and time as given by:

$$a_T = \frac{\Delta v}{\Delta t}$$

Now, $$\Delta v = \sqrt{(0.3)^2 + (0.4)^2} = \sqrt{0.25} = 0.5\ ms^{-1}$$

$\therefore \qquad a_T = \frac{\Delta v}{\Delta t} = \frac{0.5}{1} = 0.5\ ms^{-2}$

Illustration 44: A particle is moving in a circular orbit with a constant tangential acceleration. After 2 seconds from the beginning of motion, angle between the total acceleration vector and the radius R becomes 45°. What is the angular acceleration of the particle?

[JEE MAIN]

Sol: In the adjoining figure are shown the total acceleration vector $\vec{a}$ and its components the tangential acceleration $\vec{a}_T$ and centripetal acceleration $\vec{a}_c$ are shown. These two components are always mutually perpendicular to each other and act along the tangent to the circle and radius respectively. Therefore, if the total acceleration vector makes an angle of 45° with the radius, both the tangential and the normal components must be equal in magnitude. Hence,

$$a_T = a_c$$

$$\Rightarrow \qquad \alpha R = \omega^2 R$$

$$\Rightarrow \qquad \alpha = \omega^2 \qquad \text{(i)}$$

Since angular acceleration is uniform, we have,

$$\omega = \omega_0 + \alpha t$$

Substituting $\omega_0 = 0$ and t = 2 s, we have,

$$\omega = 2\alpha \qquad \text{(ii)}$$

From equation (i) and (ii), we have $\alpha = 0.25$ rad s^{-2}

EXAMPLES

PART I – JEE MAIN

Example 1: A person in his morning walk moves on a semi-circular track of radius 40 m. Find the distance travelled and the displacement, when he starts from one end of the track and reaches the other end.

Sol: Distance is length of the path travelled. Displacement is the vector from initial point to final point.

The distance covered = length of the semi-circular track = $\pi R = 3.14 \times 40$ m = 125.6 m

Displacement = Final position - Initial position = Diameter of semi-circular track = $2R = 2 \times 40 = 80$ m

Initial point to final point gives the direction of displacement.

Example 2: A man walks 2.5 km from his house to the market on a straight road with a speed of 5 km h^{-1}. He instantly turns back home with a speed of 7.5 km h^{-1} finding the market closed. Calculate:

(a) Magnitude of Average Velocity
(b) Average speed of the man over the interval of time:

(i) 0 to 30 min
(ii) 0 to 50 min
(iii) 0 to 40 min

Sol: Average speed is distance covered divided by time taken. Distance is length of the path travelled. Average velocity is displacement divided by time taken. Displacement is the vector from initial point to final point.

Distance between market and home = 2.5 km

Speed of man from home to market = 5 km h^{-1}

∴ Time taken by the man to reach the market, $t_1 = \frac{\text{Distance}}{\text{Speed}} = \frac{2.5}{5} = \frac{1}{2}$ h = 30 min

Speed of man during his return = 7.5 km h^{-1}

∴ Time taken by the man to return home, $t_2 = \frac{2.5}{7.5} = \frac{1}{3}$ h = 20 min

Total time taken by the man returning home = 30 + 20 = 50 min

(i) Over the interval 0 to 30 min: During this time, man goes from home to market. Therefore, displacement s = 2.5 km.

$$\text{Average Velocity} = \frac{\text{Displacement}}{\text{Time}} = \frac{2.5}{0.5} = 5 \text{ km h}^{-1}$$

$$\text{Average Speed} = \frac{\text{Distance}}{\text{Time}} = \frac{2.5}{0.5} = 5 \text{ km h}^{-1}$$

(ii) For the time interval 0 to 50 min: In 50 min the man goes from home to market and return back

∴ Displacement = zero

$$\text{Average Velocity} = \frac{\text{Displacement}}{\text{Time}} = \frac{0}{(50/60)} = 0 \text{ km h}^{-1}$$

$$\text{Average Speed} = \frac{\text{Total Distance}}{\text{Time}} = \frac{2.5 + 2.5}{(50/60)} = 6 \text{ km h}^{-1}$$

(iii) During the time interval 0 to 4 min: During first 30 min man goes home to market converting a distance 2.5 km in next 10 min the man is in path from market to home and comes a distance,

$$= 7.5 \times \frac{10}{60} = 1.25 \text{ km}$$

Displacement = 2.5 - 1.25 = 1.25 km

Distance = 2.5 + 1.25 = 3.75 km

$$\text{Average Velocity} = \frac{\text{Displacement}}{\text{Time}} = \frac{1.5}{(40/60)} = 1.875 \text{ km h}^{-1}$$

$$\text{Average Speed} = \frac{\text{Distance}}{\text{Time}} = \frac{3.75}{(40/60)} = 5.625 \text{ km h}^{-1}$$

Example 3: A particle moving with an initial velocity 2.5 ms^{-1} along the positive x direction accelerates uniformly at the rate 0.50 ms^{-2}.

(i) Find the distance travelled in the first 2 sec
(ii) Calculate the time taken to reach the velocity of 7.5 ms^{-1}
(iii) Calculate the distance travelled in reaching the velocity 7.5 ms^{-1}

Sol: This is the case of motion with uniform acceleration. Use the three equations of motion with uniform acceleration.

(i) We have, $x = ut + \frac{1}{2}at^2 = (2.5)(2) + \frac{1}{2}(0.5)(2)^2 = 5 + 1 = 6\text{m}$

Since the particle does not turn back it is also the distance travelled.

(ii) We have, $v = u + at$

$\Rightarrow \quad 7.5 = 2.5 + (0.50)t$

$\Rightarrow \quad t = \frac{7.5 - 2.5}{0.5} = 10 \text{ s}$

(iii) We have, $v^2 = u^2 + 2ax$

$\Rightarrow \quad (7.5)^2 = (2.5)^2 + 2(0.50)x$

$\Rightarrow \quad x = \frac{7.5 - 2.5}{2 \times 5.0} = 50 \text{ m}$

Example 4: A particle is projected vertically upwards with velocity 40 ms^{-1}. Find the displacement and distance travelled by the particle in 2s, 4s and 6s. [Take g = 10 ms^{-2}]

Sol: Distance covered is equal to displacement if the object moves in a straight line and there is no change in direction of motion. If direction of motion changes, distance should be calculated separately for different parts of the path. Here, due to upward motion, u is positive and due to downward motion, a is negative. Velocity becomes zero at maximum height

Time taken to reach maximum height, $t_0 = \frac{u}{g} = \frac{40}{10} = 4\text{s}$

(i) $t < t_0$. Therefore, distance and displacement are equal.

$\therefore \quad d = s = ut + \frac{1}{2}at^2 = (40)(2) - \frac{1}{2}(10)(4) = 60 \text{ m}$

(ii) $t = t_0$, then distance and displacement are equal.

$\therefore \quad d = s = ut + \frac{1}{2}at^2 = (40)(4) - \frac{1}{2}(10)(16) = 80 \text{ m}$

(iii) $t > t_0$. Hence, $d > s$

$\therefore \quad s = ut + \frac{1}{2}at^2 = (40)(6) - \frac{1}{2}(10)(36) = 60 \text{ m}$

And $d = \left|\frac{u^2}{2g}\right| + \frac{1}{2}|g(t - t_0)^2| = \frac{(40)^2}{(2)(10)} + \frac{1}{2}(10)(6 - 4)^2 = 100$ m

Example 5: Following information about an object's motion is given: $a = t^2$, Initial velocity = u. Find:

(i) Velocity (v) as a function of time
(ii) Displacement (x) a function of time

Sol: This is the case of motion with non-uniform acceleration. Acceleration is given as a function of time. Change in velocity can be found by integrating the expression for acceleration with respect to time. Displacement can be found by integrating the expression for velocity.

(i) $a = \frac{dv}{dt} = t^2$

$\Rightarrow$ $dv = t^2 dt$

$\Rightarrow$ $\int_u^v dv = \int_0^t t^2 dt$

$\Rightarrow$ $v = \frac{t^3}{3} + u$

(ii) $v = \frac{dx}{dt} = \frac{u + t^3}{3}$

$\Rightarrow$ $dx = \left(\frac{u + t^3}{3}\right) dt$

$\Rightarrow$ $\int_0^x dx = \int_0^t u dt + \int_0^t \frac{t^3}{3} dt$

$\Rightarrow$ $x = ut + \frac{t^4}{12}$

Example 6: The position of an object moving along x-axis is given by $x = 8.0 + 2.0t^2$, where x is in meter and t is in second. Calculate:

(i) the velocity at t = 0 and t = 2.0 sec
(ii) average velocity between 2.0 sec and 4.0 sec

Sol: Here the position is given as a function of time. Differentiate this expression w.r.t time to get velocity as a function of time. Average velocity is displacement divided by time taken.

(i) Velocity $= v = \frac{dx}{dt} = \frac{d(8.0 + 2.0t^2)}{dt} = 0 + (2.0)(2t) = 4t$

$\therefore$ Velocity at 0 sec = $4 \times 0 = 0$

$\therefore$ Velocity at 2 sec = $4 \times 2 = 8\ ms^{-1}$

(ii) Average Velocity $= v_{avg} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t} = \frac{(x)_{t=4} - (x)_{t=2}}{4 - 2} = \frac{[8 + 2(4)^2] - [8 + 2(2)^2]}{2} = \frac{40 - 16}{2} = 12\ ms^{-1}$

Example 7: The motion of two bodies A and B represented by two straight lines drawn on the same displacement-time graph, make angles 30° and 60° with time axis, respectively. Which body possesses greater velocity? What is the ratio of their velocities?

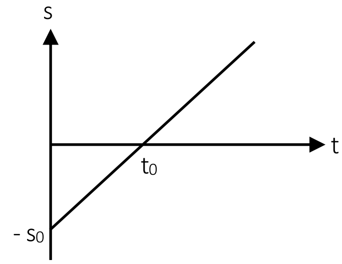
Sol: The slope of the displacement time graph at an instant gives the velocity at that instant.

The velocity of body is equal to the slope of displacement time graph. Therefore, the line having greater slope has greater velocity, i.e. the body B has greater velocity.

Ratio of their velocities,

$$\frac{v_A}{v_B} = \frac{\tan 30°}{\tan 60°} = \frac{1/\sqrt{3}}{\sqrt{3}} = \frac{1}{3}$$

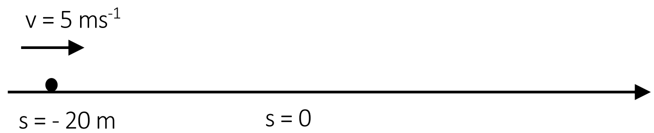
Example 8: Displacement–time graph of a particle moving in a straight line is as shown in Figure. State whether the motion is accelerated or not. Describe the motion in detail. Given s_0 = 20 m and t_0 = 4 s.



Sol: The slope of the displacement time graph at an instant gives the velocity at that instant. If the slope is constant, the velocity is uniform (zero acceleration). If the slope changes with time the motion is accelerated. Slope of s–t graph is constant. Hence, velocity of particle is constant. Further at time t = 0, displacement of the particle from the mean position is - s_0 i.e. - 20 m.

Velocity of particle, $v = \text{slope} = \frac{s_0}{t_0} = \frac{20}{4} = 5\ ms^{-1}$

Motion of the particle is as shown in figure. At t = 0, particle is at - 20 m and has a constant velocity of 5 ms^{-1}. At t_0 = 4 sec, particle will pass through its mean position.



Example 9: Abhishek is moving with velocity $5\hat{i} + 3\hat{j}$ and Amit with velocity $2\hat{i} + 4\hat{j} - 3\hat{k}$. Find:

(i) Relative velocity of Abhishek with respect to Amit $(\vec{v}_{12})$
(ii) Relative velocity of Amit with respect to Abhishek $(\vec{v}_{21})$

Sol: Relative velocity of body 1 with respect to body 2 is obtained by vector sum of the velocity of body 1 and the negative of the velocity of body 2.

Velocity of Abhishek $(\vec{v}_1) = 5\hat{i} + 3\hat{j}$

Velocity of Amit $(\vec{v}_2) = 2\hat{i} + 4\hat{j} - 3\hat{k}$

(i) $\vec{v}_{12} = \vec{v}_1 - \vec{v}_2 = (5\hat{i} + 3\hat{j}) - (2\hat{i} + 4\hat{j} - 3\hat{k}) = 7\hat{i} - \hat{j} + 3\hat{k}$

(ii) $\vec{v}_{12} = -\vec{v}_{21} = -\ 7\hat{i} + \hat{j} - 3\hat{k}$

Example 10: Two trains, each of length 100 m, move in opposite direction along parallel lines at speeds 60 km h^{-1} and 30 km h^{-1}, respectively. If their accelerations are 30 cm s^{-2} and 20 cm s^{-2}, respectively, then find the time they take to pass each other.

Sol: This problem is best solved in the reference frame of one of the trains. Find the initial relative velocity of one train with respect to the other, the relative acceleration of one train with respect to the other and the relative displacement of one train with respect to other as they pass each other. Use the second equation of motion with constant acceleration to find the required time.

The relative displacement of the trains, s =100 + 100 = 200

The initial velocity u of one train relative to the other train = (60 + 30) km $h^{-1} = 90 \times \frac{5}{18}\ ms^{-1} = 25\ ms^{-1}$

Relative acceleration, a = (30 + 20) cm s^{-2} = 0.5 ms^{-2}

If "t" is the time taken to cross each other,

$$s = ut + \frac{1}{2}at^2$$

$$\Rightarrow \quad 200 = 25t + \frac{1}{2}(0.5)t^2$$

$$\Rightarrow \quad 0.5t^2 + 50t - 400 = 0$$

$\Rightarrow \quad t = \dfrac{-50 \pm \sqrt{2500 + (4)(400)(0.5)}}{(2)(0.5)} = -50 \pm \sqrt{3300}$

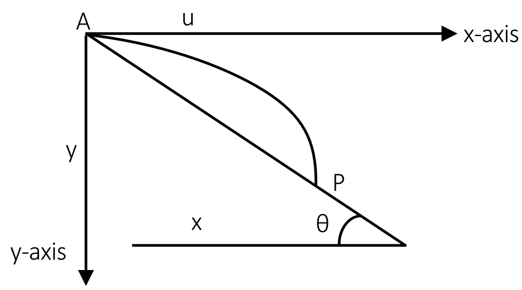
As negative t is ignored,

$\therefore \quad t = -50 + \sqrt{3300} = -50 + 57.44$

$\Rightarrow \quad t = 7.44$ s

Example 11: A particle is projected horizontally with a speed u from the top of a plane inclined at an angle θ with the horizontal. How far from the point of projection will the particle strike the plane?

Sol: Take the x-axis parallel to the horizontal. Take the y-axis along the vertical. Along x-axis velocity is uniform. Along y-axis initial velocity is zero and acceleration is uniform. Take, X–Y axes as shown in figure. Suppose that the particle strikes the plane at point P with coordinates (x and y). Consider the motion between A and P.



Motion in x-direction: Initial velocity = u

Acceleration = 0, x = ut (i)

Motion in y-direction: initial velocity = 0

Acceleration = g, $y = \frac{1}{2}gt^2$ (ii)

Eliminating t from (i) and (ii)

Thus, $y = \frac{1}{2}g\frac{x^2}{u^2}$

Also, $y = x\tan\theta$

Thus, $y = \frac{1}{2}g\frac{x^2}{u^2} = x\tan\theta$

$\Rightarrow \quad x = 0$ or $x = \dfrac{2u^2\tan\theta}{g}$

Then, $y = x\tan\theta = \dfrac{2u^2\tan^2\theta}{g}$

The distance $AP = l = \sqrt{x^2 + y^2} = \dfrac{2u^2}{g}\tan\theta\sqrt{1 + \tan^2\theta} = \dfrac{2u^2}{g}\tan\theta\sec\theta$

Examples 12: A projectile is projected at an angle 60° from the horizontal with a speed of $(\sqrt{3} + 1)\ ms^{-1}$. The time (in seconds) after which the inclination of the projectile with horizontal becomes 45° is:

Sol: Take the x-axis along the horizontal. Take the y-axis vertically upwards. Along x-axis velocity is uniform. Along y-axis initial velocity is positive and acceleration is uniform and negative.

Let "u" and "v" be the speed at the two specified angles. The initial components of velocities in horizontal and vertical directions are:

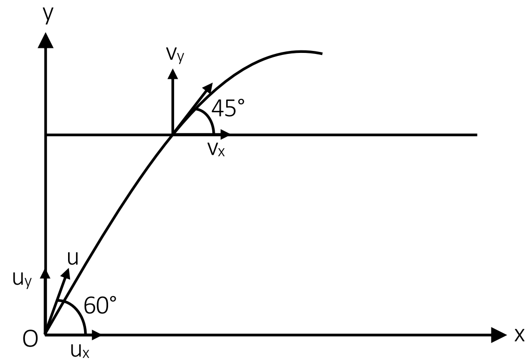
$u_x = u\cos 60°$ and $u_y = u\sin 60°$

Similarly, the components of velocities, when the projectile makes an angle 45 with horizontal and vertical directions are:

$v_x = v\cos 45°$ and $v_y = v\sin 45°$

But we know that horizontal component of velocity remains unaltered during motion. Hence,

$$v_x = u_x \Rightarrow v\cos 45° = u\cos 60° \Rightarrow v = \frac{u\cos 60°}{\cos 45°}$$



Here, we know initial and final velocities in vertical direction. We can apply 'v = u + at' in vertical direction to know the time as required:

$$v\sin 45° = u\sin 60° - gt$$

$$\Rightarrow \quad t = \frac{u\sin 60° - u\left(\frac{\cos 60°}{\cos 45°}\right)\sin 45°}{g}$$

$$\Rightarrow \quad t = 0.1\text{ s}$$

Example 13: A projectile is at an angle " θ " from the horizontal at the speed "u". If an acceleration of "g/2" is applied to the projectile due to wind in horizontal direction, then find the new time of flight, maximum height and horizontal range.

Sol: Take the x-axis along the horizontal. Take the y-axis vertically upwards. Along x-axis initial velocity is positive and acceleration is uniform and positive. Along y-axis initial velocity is positive and acceleration is uniform and negative. The acceleration due to wind affects only the motion in horizontal direction. It would, therefore, not affect attributes like time of flight or maximum height that results exclusively from the consideration of motion in vertical direction. The generic expressions of time of flight, maximum height and horizontal range of flight with acceleration are given as under:

$$T = \frac{2u_y}{g},\ H = \frac{u_y^2}{2g} = \frac{gT^2}{4},\ R = \frac{u_x u_y}{g}$$

The expressions above revalidate the assumption made in the beginning. We can see that it is only the horizontal range that depends on the component of motion in horizontal direction. Now, considering accelerated motion in horizontal direction, we have:

$$x = R' = u_x T + \frac{1}{2}a_x T^2$$

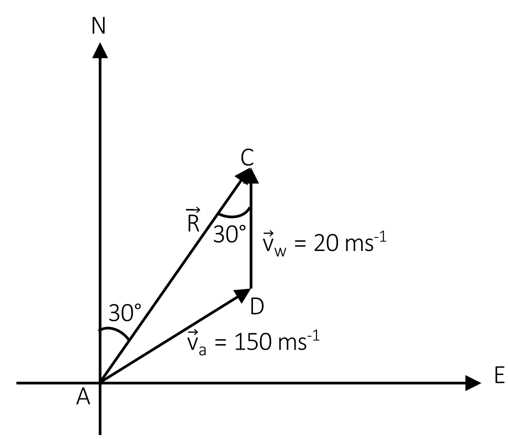
$$\Rightarrow \quad R' = u_x T + \frac{1}{2}\left(\frac{g}{2}\right)T^2$$

$$\Rightarrow \quad R' = R + H$$

Example 14: An airplane has to go from a point A to another point B, 500 km away due 30° east of north. A wind is blowing due north at a speed of 20 ms^{-1}. The air speed of the plane is 150 ms^{-1}.

(i) Find the direction in which the pilot should head the plane to reach the point B.
(ii) Find the time taken by the plane to go from A to B.

Sol: The vector sum of the velocity of the airplane with respect to the wind and the velocity of the wind with respect to ground is equal to velocity of the aircraft with respect to ground. This net velocity should be in the direction A to B. In the resultant direction R, the plane reaches the point B.



Velocity of wind $\vec{v}_w = 20\ ms^{-1}$

Velocity of aero plane a $\vec{v}_a = 150\ ms^{-1}$

In ΔACD, according to the sine formula

$$\therefore \quad \frac{20}{\sin A} = \frac{150}{\sin 30°}$$

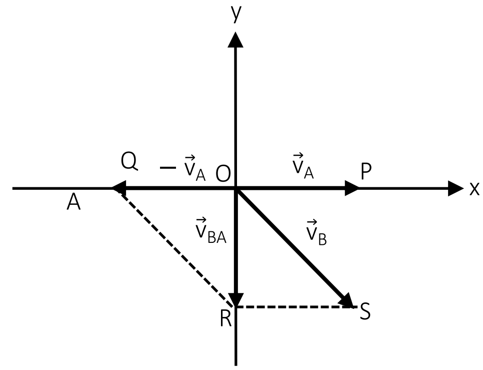
$$\Rightarrow \quad A = \sin^{-1}\left(\frac{1}{15}\right)$$

(i) The direction is $\sin^{-1}(1/15)$ east of the line AB.

(ii) $\sin^{-1}(1/15) = 3°48' \Rightarrow 30° + 3°48' = 33°48'$

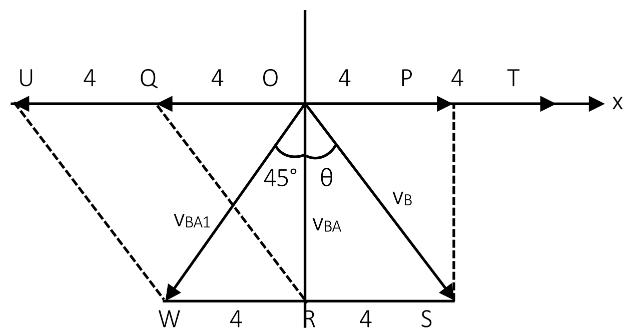
$$R = \sqrt{150^2 + 20^2 + 2(150)(20)\cos 33°48'} = 167\ ms^{-1}$$

$$T = \frac{s}{v} = \frac{500000}{167} = 2994\ s = 50\ min$$



Example 15: Rain drops appear to fall in vertical direction to a person, who is walking at a velocity 4 ms^{-1} in a given direction. When the person doubles his velocity in the same direction, the rain drops appear to come at an angle of 45° from the vertical. Find the speed of the rain drop.

Sol: Velocity of rain with respect to the person is equal to the vector sum of the velocity of rain with respect to ground and the negative of the velocity of person with respect to ground. The direction of velocity of rain with respect to person is known in each case. Assume a direction of velocity of rain with respect to ground and draw the vector diagrams for velocity of rain with respect to person for both the cases. This is a slightly tricky question. Readers may like to visualize the problem and solve on their own before going through the solution given here. Let us consider the situation under two cases. Here, only the directions of relative velocities in two conditions are given. The Figure on the left represents initial situation. Here, the vector OP represents velocity of the person ($\vec{v}_A$), OR represents relative velocity of rain drop with respect to person ($\vec{v}_B$); OS represents velocity of rain drop.



The given figure represents situation when person starts moving with double velocity. Here, the vector OT represents velocity of the person ($\vec{v}_{A1}$), OW represents relative velocity of rain drop with respect to person ($\vec{v}_{BA1}$). We should note that velocity of rain ($\vec{v}_B$) drop remains the same and as such, it is represented by OS represents as before.

According to the question, we are required to know the speed of the rain drop. It means that we need to know the angle "θ" and the side OS, which is the magnitude of velocity of rain drop. It is intuitive from the situation that it would help if we consider the vector diagram and carry out geometric analysis to find these quantities. For this, we substitute the vector notations with known magnitudes as shown hereunder.

We note here that, WR = UQ = 4 ms^{-1}

Clearly, triangles ORS and ORW are congruent as two sides and one enclosed angle are equal.

WR = RS = 4 ms^{-1}, OR = OR, $\angle ORW = \angle ORS = 90°$

Hence, $\angle WOR = \angle SOR = 45°$

In triangle ORS, $\sin 45° = \frac{RS}{OS} \Rightarrow OS = \frac{RS}{\sin 45°} = 4\sqrt{2}\ ms^{-1}$

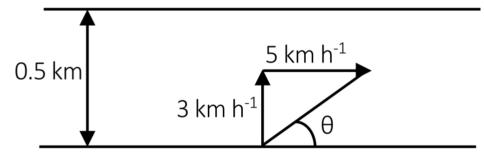
Example 16: A swimmer wishes to cross a 500 m wide river flowing at 5 km h^{-1}. His speed with respect to water is 3 km h^{-1}.

(i) If he heads in a direction making an angle θ with the flow, find the time he takes to cross the river.
(ii) Find the shortest possible time to cross the river.

Sol: Time taken to cross the river will depend on the component of velocity of swimmer which is perpendicular to the river flow. For shortest time this component should be maximum, i.e. θ = 90°.

(i) The vertical component 3sin θ takes him to the opposite side.

Distance = 0.5 km, Velocity = 3sinθ km h^{-1}



$$\text{Time} = \frac{\text{Distance}}{\text{Velocity}} = \frac{0.5}{3\sin\theta} = \frac{10}{\sin\theta}\ \text{min}$$

(ii) Here vertical component of the velocity, i.e., 3 km h^{-1} takes him to the opposite side (θ = 90°).

$$\text{Time} = \frac{\text{Distance}}{\text{Velocity}} = \frac{0.5}{3} = 0.16\ \text{h} = 10\ \text{min}$$

Example 17: Two tall buildings are 200 m apart. With what speed must a ball be thrown horizontally from a window of one building 2 km above the ground so that it will enter a window 40 m from the ground in the other?

Sol: The time taken by ball to fall from the height of 2000 m to the height of 40 m (with zero initial velocity in vertical direction) should be equal to the time taken by ball to cover a horizontal distance of 200 m with constant velocity in horizontal direction.

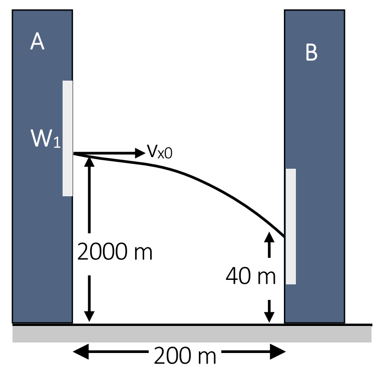


Figure shows the conditions of the problem. Here, A and B are the two tall buildings having windows W_1 and W_2, respectively. The window W_1 is 2 km (= 2000 m) above the ground while window W_2 is 40 m above the ground. We want to throw the ball from window W_1 with such a horizontal speed (v_{x0}) so that it enters the window W_2. Note that the horizontal range of the ball is R = 200 m. Let t sec be the time taken by the ball to reach from window W_1 to window W_1. This time will depend upon the vertical motion (downward) alone.

For vertical motion: h = 2000 - 40 = 1960 m, g = 9.8 ms^{-2}, $v_{y0} = 0$

$$\therefore \qquad t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2(1960)}{9.8}} = 20\ \text{s}$$

$$\text{Now} = \frac{R}{t} = \frac{200}{20} = 10\ \text{ms}^{-1}$$

Example 18: A particle moves in a circle of radius 20 cm. Its linear speed is given by v = 2t where t is in second and v in metre/second. Find the radial and tangential acceleration at t = 3 s.

Sol: Radial acceleration depends on the square of instantaneous speed and the radius. Tangential acceleration is equal to the rate of change of instantaneous speed.

The linear speed at t = 3 s is v = 2t = 6 ms^{-1}

The radial acceleration at t = 3 s is $a_r = \frac{v^2}{r} = \frac{36}{0.2} = 180\ \text{ms}^{-2}$

The tangential acceleration is $a_t = \frac{dv}{dt} = \frac{d(2t)}{dt} = 2\ \text{ms}^{-2}$

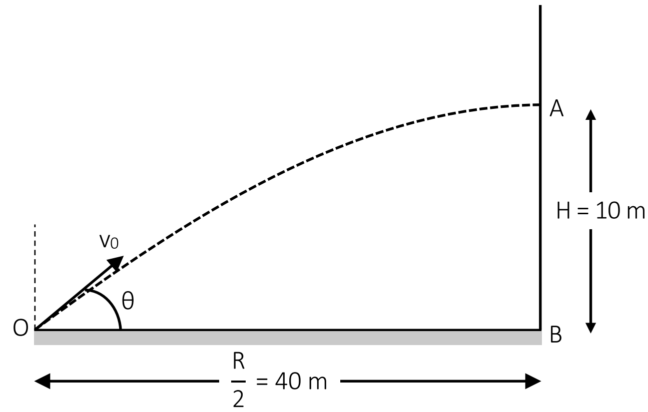
Example 19: A boy wants to throw a letter wrapped over a stone to his friend across the street 40 m wide. The boy's window is 10 m below friend's window. How should he throw the ball?

Sol: We assume that the boy throws the ball such that the maximum height attained by the ball is H = 10 m. This implies that the range of the ball is R = 40 × 2 = 80 m. Thus, from the formula of H and R we can find the values of initial velocity and the angle of projection. Figure shows the conditions of the problem. The boy's window is at O and friend's window is at A. Let the boy throw the stone with a velocity v_0 making an angle θ with the horizontal so as to enter the window at A. The stone will follow the parabolic path with A as the highest point on the trajectory of stone.

R = 2 × 40 = 80 m

Motion in a plane:

$$R = \frac{v_0^2 \sin 2\theta}{g} = \frac{v_0^2 2 \sin\theta \cos\theta}{g} \text{ and } H = \frac{v_0^2 \sin^2\theta}{2g}$$



$$\Rightarrow \quad \frac{H}{R} = \frac{\tan\theta}{4}$$

$$\Rightarrow \quad \tan\theta = 4 \times \frac{H}{R} = 4 \times \frac{10}{80} = \frac{1}{2}$$

$$\Rightarrow \quad \theta = 26.56°$$

Maximum height attained, H = 10 m

Now, the projection velocity v_0 can be found by substituting the value of θ in formula for H.

$$H = \frac{v_0^2 \sin^2\theta}{2g}$$

$$\Rightarrow \quad v_0 = \sqrt{\frac{2gH}{\sin^2\theta}} = \sqrt{\frac{2 \times 9.8 \times 10}{\sin^2 26.56°}} = 31.3 \text{ ms}^{-1}$$

Example 20: A body is projected with a velocity of 40 ms^{-1}. After 2 s, it crosses a vertical pole of height 20.4 m. Calculate the angle of projection and the horizontal range.

Sol: Use second equation of motion with constant acceleration in vertical direction. Let θ be the angle of projection. For vertical motion,

$$h = (v_0 \sin\theta)t - \frac{1}{2}gt^2$$

$$\Rightarrow \quad 20.4 = 80 \sin\theta - 19.6$$

$$\Rightarrow \quad \sin\theta = \frac{20.4 + 19.6}{80} = \frac{40}{80} = \frac{1}{2}$$

$$\Rightarrow \quad \theta = 30°$$

Horizontal Range, $R = \frac{v_0^2 \sin 2\theta}{g} = \frac{(40)^2 \times \sin 60°}{9.8} = 141.1 \text{ m}$

PART II – JEE ADVANCED

Example 1: Two cars started simultaneously towards each other from towns A and B which are 480 km apart. It took first car travelling from A to B 8 hours to cover the distance and second car travelling from B to A 12 hours. Determine the distance (in km) from town A where the cars meet. Assuming that both the cars travelled with constant speed.

Sol: This problem is best solved in the reference frame of one of the cars. Find the initial relative velocity of one car with respect to the other. The time elapsed before the cars meet is equal to the initial distance between the two cars divided by the relative velocity.

Velocity of car from A = $\frac{480}{8}$ = 60 km h^{-1}

Velocity of car from B = $\frac{480}{12}$ = 40 km h^{-1}

∴ $t = \frac{480}{60 + 40} = 4.8$ h

The distance $s = v_A \times t = 60 \times 4.8 = 288$ km

Example 2: An engine driver running a train at full speed suddenly applies brakes and shuts off steam. The train then travels 24 m in the first second and 22 m in the next second. Assuming that the brakes produce a constant retardation, find

(i) Original speed of the train
(ii) The time elapsed before it comes to rest
(iii) The distance travelled during the interval
(iv) If the length of the train is 44 m, find the time that the train takes to pass an observer standing at a distance 100 m ahead of the train at the time when the brake was applied.

Sol: The retardation of train is constant, so we can use the equations of motion with uniform acceleration. The acceleration is taken with a negative sign.

(i) The distance covered by the body in nth second,

$$s_n = u + \frac{a}{2}(2n - 1)$$

∴ $s_1 = 24 = u - \frac{a}{2}(2 - 1) = u - \frac{a}{2}$

And, $s_2 = 22 = u - \frac{a}{2}(4 - 1) = u - \frac{3a}{2}$

Solving the above two equations, we get a = 2 ms^{-2} and u = 25 ms^{-1}

(ii) Time t taken by the train before coming to rest,

$v = u – at$

⇒ $0 = 25 – 2t$

⇒ $t = 12.5$ s

(iii) If s is the distance before the train comes to rest i.e. v = 0, then,

$0^2 = u^2 - 2as$

⇒ $s = \frac{u^2}{2a} = \frac{(25)^2}{4} = 156.25$ m

(iv) The time t taken by the train to cover a distance of 100 m is given by,

$$s = ut - \frac{1}{2}at^2$$

$\Rightarrow$ $$100 = 25t - \frac{1}{2}(2)(t)^2$$

$\Rightarrow$ $$t^2 - 25t + 100 = 0$$

$\Rightarrow$ $$(t - 20)(t - 5) = 0$$

$\Rightarrow$ $$t = 20 \text{ or } t=5,$$

t = 20 is not possible as the train takes only 12.5 second to stop. Therefore t = 5 second.

Time t' taken by the train to cover a distance of 100 m plus length of the train, i.e., 44 m, is given by,

$$s = 100 + 44 = ut' - \frac{1}{2}at'^2$$

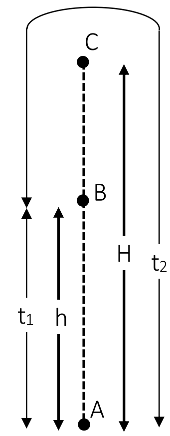
$\Rightarrow$ $$t'^2 - 25t' + 144 = 0$$

$\Rightarrow$ $$(t' - 16)(t' - 9) = 0$$

$\Rightarrow$ $$t' = 16 \text{ s or } t' = 9 \text{ s}$$

$\therefore$ Time taken by the train to pass the observer = 9 - 5 = 4 s

Example 3: A particle is projected vertically upwards from a point A on the ground. It takes a time t_1 to reach a point B at a height h above the ground as it continues to move, it takes a further time t_2 to reach the ground. Find:



(i) The height h
(ii) The maximum height reached
(iii) The velocity of the particle at half the maximum height.

Sol: (i) Find the initial velocity u in terms of t_1 and t_2. Use the equations of motion with uniform acceleration. The acceleration due to gravity is taken with a negative sign.

Let u be initial velocity. Total time of flight from A to B and from B to C to A is, $t_1 + t_2$, then,

$$t_1 + t_2 = \frac{2u}{g}$$

$\Rightarrow$ $$u = \frac{g(t_1 + t_2)}{2}$$

Also, $$h = ut_1 - \frac{1}{2}gt_1^2 = \frac{g}{2}(t_1 + t_2)t_1 - \frac{1}{2}gt_1^2 = \frac{gt_1t_2}{2}$$

(ii) Maximum height reached, AC = H = $\frac{u^2}{2g} = \frac{g^2(t_1 + t_2)^2}{(4)(2g)} = \frac{g(t_1 + t_2)^2}{8}$

(iii) Let v be velocity at height $\frac{H}{2}$, then,

$$v^2 = u^2 - 2g\frac{H}{2} = u^2 - gH = \frac{g^2(t_1 + t_2)^2}{4} - \frac{g(t_1 + t_2)^2}{8} = \frac{g(t_1 + t_2)^2}{8}$$

$\Rightarrow$ $$v = \frac{g}{2\sqrt{2}}(t_1 + t_2)$$

Example 4: If $v(s) = s^2 + s$ where s is displacement. Find acceleration when displacement is 1 m.

Sol: Differentiate the expression for velocity with respect to time to get the expression for acceleration.

We know, $a = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\left(\frac{dv}{ds}\right)$

$\therefore \quad \frac{dv}{ds} = 2s + 1$

Now, $\left(\frac{dv}{ds}\right)_{s=1} = 3$ m

Also, $v(s) = s^2 + s$

$\therefore \quad v(1) = (1)^2 + 1 = 2$

$\therefore \quad a(s = 1) = 2 \times 3 = 6\ ms^{-2}$

Example 5: A point mass moves along a straight line with a deceleration n which is equal to $K\sqrt{v}$ where K is a positive constant and v is the velocity of the particle. The velocity of the point mass at t = 0 is equal to v_0. Find the distance it will travel before it stops and the time it will take to cover this distance.

Sol: In the expression for acceleration separate the variables and integrate to get the desired quantity.

Acceleration $= \frac{dv}{dt} = -K\sqrt{v}$

$\Rightarrow \quad -\frac{1}{\sqrt{v}}dv = kdt$

Let t_0 be the time which the particle takes to come to a stop. Then,

$$\int_0^{t_0} Kdt = -\int_{v_0}^{0} v^{-\frac{1}{2}}dv$$

$\Rightarrow \quad t_0 = \frac{2\sqrt{v_0}}{k}$

Also, $\quad v\frac{dv}{ds} = -K\sqrt{v}$

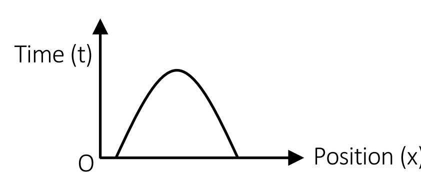
$\Rightarrow \quad \sqrt{v}dv = -kds$

Let s_0 be the distance covered when the velocity decrease from v_0 to zero. Then,

$$\int_{v_0}^{0} \sqrt{v}dv = -\int_0^{s_0} kds$$

$\Rightarrow \quad s_0 = \frac{2v_0^{3/2}}{3k}$

Example 6: Is the variation of position, shown in the figure observed in nature?



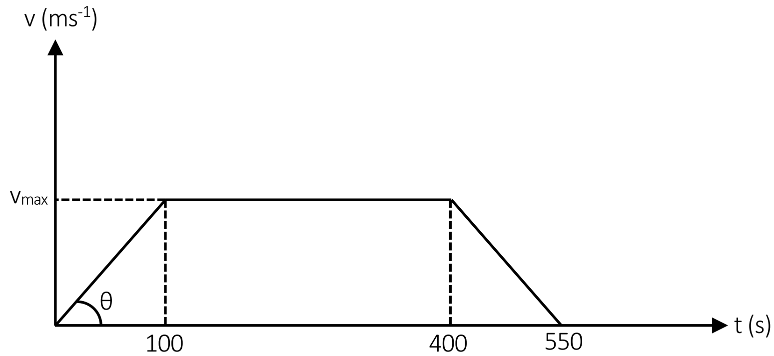
Sol: Time never decreases in a reference frame. No, since with increase of position x, time first increase and then decrease, which is impossible (Time always increase)

Example 7: A particle start from rest with constant acceleration for 100 s, then move with constant velocity for 5 min. Finally, particle retards uniformly and come to rest in 150 s.

(i) Draw v-t graphs
(ii) If total distance travelled by the particle is 4250 m then find maximum speed.
(iii) Also, find the value of acceleration and retardation.

Sol: Area under the v-t graph in the given time interval is equal to displacement of the particle in the given time interval.

(i)



(ii) Distance travelled = Area under v-t graph = $\left[\frac{1}{2}(100)v_{max}\right] + [(400 - 100)(v_{max})] + \left[\frac{1}{2}(550 - 400)v_{max}\right]$

$\Rightarrow \quad 4250 = \frac{1}{2}(100)v_{max} + 300v_{max} + \frac{1}{2}(150)v_{max}$

$\Rightarrow \quad v_{max} = 10\ ms^{-1}$

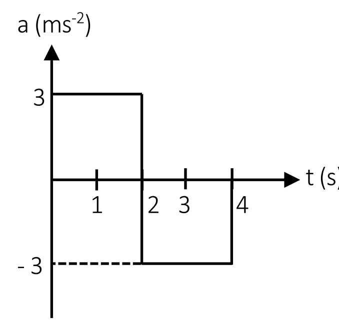
(iii) $\quad a = \frac{\Delta v}{\Delta t}$

For t = 0 to 100 s, $\quad a = \frac{10 - 0}{100 - 0} = 0.1\ ms^{-2}$

For t = 100 to 400 s, $\quad a = \frac{10 - 10}{400 - 1000} = 0$

For t = 400 to 550 s, $\quad a = \frac{0 - 10}{550 - 400} = -0.06\ ms^{-2}$

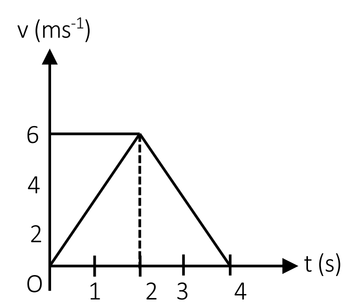
Example 8: A particle starts from rest at time t = 0 and undergoes acceleration a, as shown in the figure.



(i) Draw a neat sketch showing the velocity of the particle as a function of time during the interval 1 to 4 seconds, indicating each second on the abscissa.
(ii) Draw a neat sketch showing the displacement of the particle as a function of time during 0 to 2 second. In both the cases, explain the various steps.

Sol: Area under the a-t graph in the given time interval is equal to the change in velocity of the particle in the given time interval. Thus v-t graph can be plotted if the initial velocity is known. Area under the v-t graph in the given time interval is equal to the displacement of the particle in the given time interval.

(i) The velocity is given by the area enclosed during the time interval and the velocity is constant from 0 to 2 sec.



At t = 1 sec, Velocity = 3 ms^{-1}

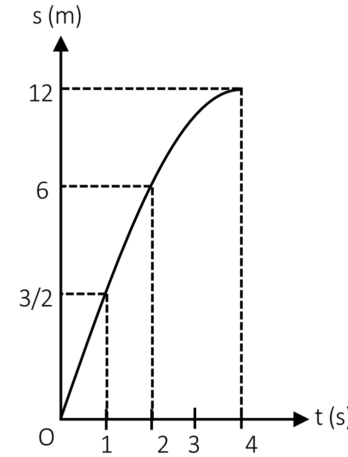
At t = 2 sec, Velocity = 6 ms^{-1}

At this instant the acceleration becomes negative, so the velocity starts decreasing uniformly.

At t = 3 sec, Velocity = 6 - 3 = 3 ms^{-1}

At t = 4 sec, Velocity = 6 - 6 = 0 ms^{-1}

(ii) The particle starts from rest, and the acceleration a is constant from 0 to 2 sec.



Since $S = \frac{1}{2}at^2$, the graph between S and t will be parabola.

At $t = \frac{1}{2}$ s, $s_{1/2} = \frac{1}{2}(3)\left(\frac{1}{2}\right)^2 = \frac{3}{8}$ m

At t = 1 s, $s_1 = \frac{1}{2}(3)(1)^2 = \frac{3}{2}$ m

At $t = \frac{3}{2}$ s, $s_{3/2} = \frac{1}{2}(3)\left(\frac{3}{2}\right)^2 = \frac{27}{8}$ m

At t = 2 s, $s_2 = \frac{1}{2}(3)(2)^2 = 6$ m

Beyond t = 4s, the acceleration becomes negative, the curvature of the graph becomes opposite at this instant. In the interval between t = 2s and t = 3s,

Distance travelled = $(6)(1) - \frac{1}{2}(3)(1)^2 = 4.5$ m

At t = 3 s, $s_3 = 6 + 4.5 = 10.5$ m

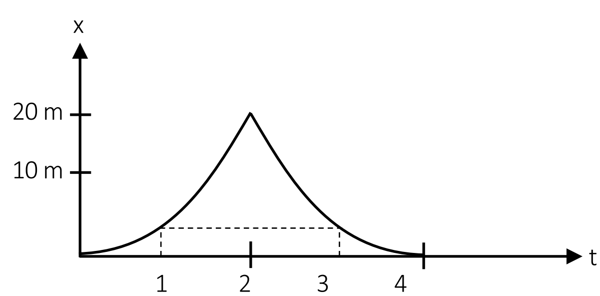
At t = 4 s, $s_4 = 6 + \left[(6)(2) - \frac{1}{2}(3)(2)^2\right] = 12$ m

Example 9: A ball is dropped from a height of 19.6 m above the ground. It rebounds from the ground and raises itself up to the same height. Take the starting point as the origin and vertically downward as the positive X-axis. Draw approximate plots of x versus t, v versus t and a versus t. Neglect the small interval during which the ball was in contact with the ground.

Sol: From the first equation of motion for constant acceleration, plot the v-t graph. From the second equation of motion for constant acceleration, plot the x-t graph. Since the acceleration of the ball during the contact is different from ‘g’, we have to treat the downward motion and the upward motion separately.

For the downward motion: $a = g = 9.8\ ms^{-2}$,

$$x = ut + \frac{1}{2}at^2 = 4.9t^2$$



The ball reaches the ground when x = 19.6 m. This gives t = 2 s. After that it moves up, x decreases and at t = 4 s, x becomes zero, the ball reaching the initial point.

We have,

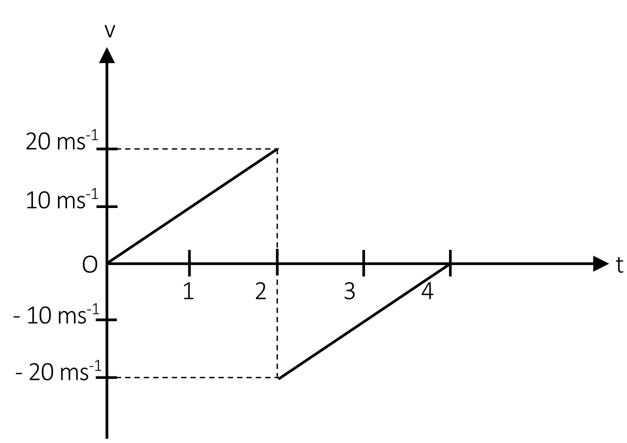
t = 0, x = 0

t = 1 s, x = 4.9 m

t = 2 s, x = 19.6 m

t = 3 s, x = 4.9 m

t = 4 s, x = 0

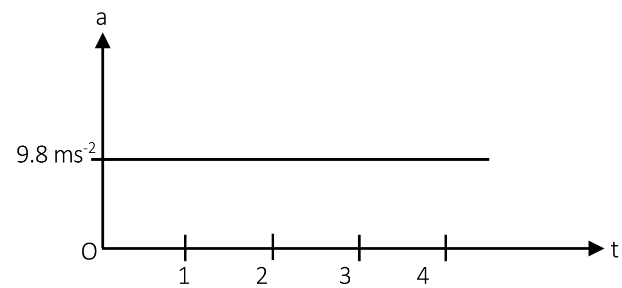


Velocity: During the first two seconds

v = u + at = 9.8t

We have,

t = 0, v = 0



t = 1 s, $v = 9.8\ ms^{-1}$

t = 2 s, $v = 19.6\ ms^{-1}$

During the next two seconds the ball goes upward, velocity is negative, magnitude decreasing and at t = 4 s, v = 0. Thus,

t = 2 s, $v = -19.6\ ms^{-1}$

t = 3 s, $v = -9.8\ ms^{-1}$

t = 4 s, v = 0.

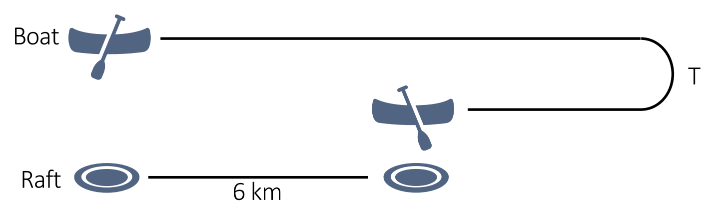
At t = 2 s there is an abrupt change in velocity from $19.6\ ms^{-1}$ to $-19.6\ ms^{-1}$. In fact, this change in velocity takes place over a small interval during which the ball remains in contact with the ground.

Acceleration: The acceleration is constant $9.8\ ms^{-2}$ throughout the motion.

Example 10: Boat is moving down the stream and crosses a raft at t = 0 sec. After 1-hour boat turns and again crosses the raft at a point 6 km from the initial position of raft. Find the velocity of river assuming duty of engine remain constant.

Note: Raft is an object which floats on the river i.e. it has zero velocity with respect to river.

Sol: Relative to the raft the boat is moving with constant speed. Distance travelled downstream relative to raft is equal to the distance travelled upstream relative to the raft. Total time of travel of boat is 2 h.



v_r = Velocity of river w.r.t ground

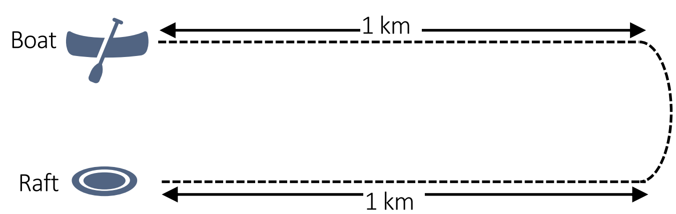
v_b = Velocity of boat w.r.t ground

Let the time taken from starting to the end = T

Velocity of raft w.r.t. ground = V_r

Distance travelled by raft in time T = 6 km

$\Rightarrow \quad v_rT = 6$ km

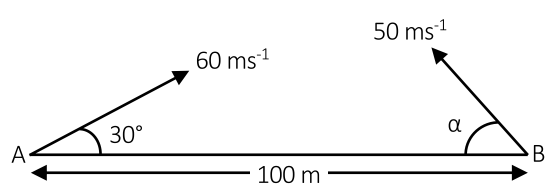


With respect to river, since boat travels for 1 h downstream and since, velocity of boat remains constant and raft doesn't move w.r.t river so, boat will again take 1 hour to reach the raft back.

$\therefore \quad T = 1 + 1 = 2$ h

$v_r(2) = 6$ km

$\Rightarrow \quad v_r = 3\ km\ h^{-1}$



Example 11: A particle A is projected with an initial velocity of $60\ ms^{-1}$ at an angle 30° to the horizontal. At the same time a second particle B is projected in opposite direction with initial speed of $50\ ms^{-1}$ from a point at a distance of 100 m from A. If the particles collide in air, find

(i) The angle of projection α of particle B
(ii) Time when the collision takes place
(iii) the distance of P from A, where collision occurs. ($g = 10ms^{-2}$)

Sol: This problem is best solved in the reference frame of one of the two particles, say particle B. The relative acceleration between the particles is zero. So, in this reference frame, the particle A moves with uniform velocity.

(i) Taking x- and y-directions as shown in the figure.

Here,

$$\vec{a}_A = -g\hat{j} \quad \text{and} \quad \vec{a}_B = g\hat{j}$$

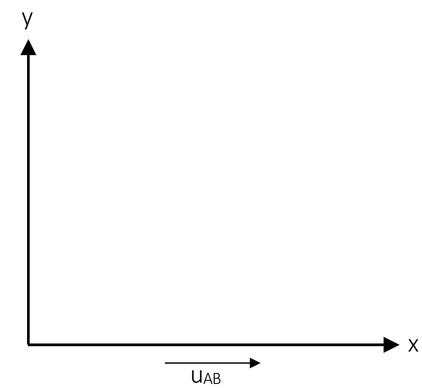
$$u_{Ax} = 60 \cos 30° = 30\sqrt{3}\ ms^{-1} \quad \text{and} \quad u_{Ay} = 60 \sin 30° = 30\ ms^{-1}$$

$$u_{Bx} = -50 \cos \alpha \quad \text{and} \quad u_{By} = 50 \sin \alpha$$

and Relative acceleration between the two is zero as $\vec{a}_A = \vec{a}_B$. Hence, the relative motion between the two is uniform. It can be assumed that B is at rest and A is moving with $\vec{u}_{AB}$. Hence, the two particles will collide, if $\vec{u}_{AB}$ is along AB. This is possible only when $u_{Ay} = u_{By}$ i.e., component of relative velocity along the y-axis should be zero.

$$30 = 50 \sin \alpha$$

$$\therefore \quad \alpha = \sin^{-1}\frac{3}{5}$$



(ii) Now,

$$|\vec{u}_{AB}| = u_{Ax} - u_{Bx} = \left(30\sqrt{3} + 50 \cos \alpha\right) = 30\sqrt{3} + (50)\left(\frac{4}{5}\right) = 30\sqrt{3} + 40\ ms^{-1}$$

Therefore, the time of collision is,

$$t = \frac{AB}{|\vec{u}_{AB}|} = \frac{100}{30\sqrt{3} + 40} = 1.09\ s$$

(iii) Distance of point P from A where collision takes place is

$$s = \sqrt{(u_{Ax}t)^2 + \left(u_{Ay}t - \frac{1}{2}gt^2\right)^2} = = \sqrt{\left(30\sqrt{3} \times 1.09\right)^2 + \left[(30 \times 1.09) - \frac{1}{2}(10 \times 1.09 \times 1.09)\right]^2} = 62.64\ m$$

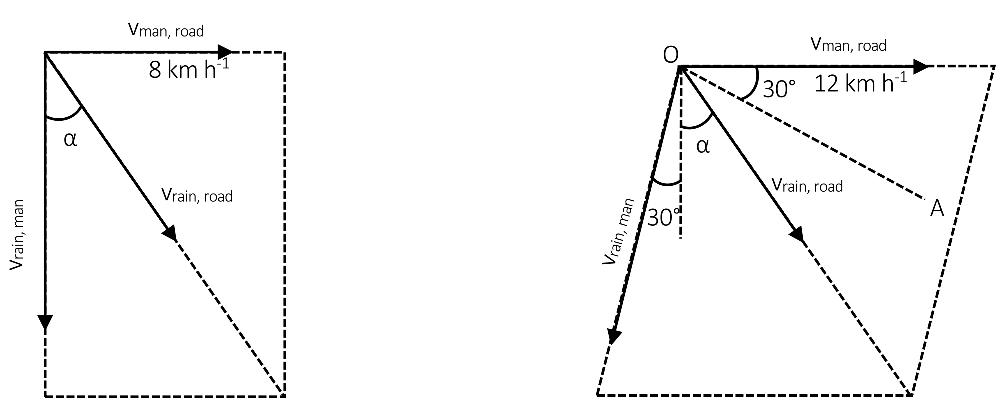
Example 12: A man running on a horizontal road at 8 km h^{-1} finds the rain falling vertically. He increases his speed to 12 km h^{-1} and finds that the drops make an angle 30° with the vertical. Find the speed and direction of the rain with respect to the road.

Sol: Velocity of rain with respect to the man is equal to the vector sum of the velocity of rain with respect to ground and the negative of the velocity of man with respect to ground. The direction of velocity of rain with respect to man is known in each case.

We have,

$$\vec{v}_{rain,road} = \vec{v}_{rain,man} + \vec{v}_{man,\ road} \qquad \text{(i)}$$

The two situations given in the problem may be represented by the following diagrams.



(a) (b)

$\vec{v}_{rain,road}$ is same in magnitude and direction in both the diagram. Taking horizontal components in equation (i) for the first diagram,

$$\vec{v}_{rain,road} \sin\alpha = 8 \text{ km h}^{-1} \qquad \text{(ii)}$$

Now, consider given figure Draw a line OA ⊥ $U_{rain,man}$ as shown. Taking components in equation (i) along the line OA, we have,

$$\vec{v}_{rain,road} \sin(30° + \alpha) = 12 \cos 30° \qquad \text{(iii)}$$

From (ii) and (iii),

$$\frac{\sin(30° + \alpha)}{\sin\alpha} = \frac{12 \times \sqrt{3}}{8 \times 2}$$

$$\Rightarrow \qquad \frac{\sin 30° \cos\alpha + \cos 30° \sin\alpha}{\sin\alpha} = \frac{3\sqrt{3}}{4}$$

$$\Rightarrow \qquad \alpha = \cot^{-1}\frac{\sqrt{3}}{2}$$

From (ii), $\vec{v}_{rain,road} = \dfrac{8}{\sin\alpha} = 4\sqrt{7} \text{ km h}^{-1}$

Example 13: Two bodies were thrown simultaneously from the same point; one, straight up, and the other, at an angle of θ = 60° to the horizontal. The initial velocity of each body is equal to $v_0 = 25 \text{ ms}^{-1}$. Neglecting the air drag, find the distance between the bodies t = 1.70 s later.

Sol: The relative acceleration of the bodies is zero. The solution of this problem becomes interesting in the frame attached with one of the bodies. Let the body thrown straight up be 1 and the other body be 2, then for the body 1 in the frame of 2 from the kinematical equation for constant acceleration (since both are moving under constant acceleration) is,

$$r_{12} = r_{0(12)} + v_{0(12)}t + \frac{1}{2}a_{12}t^2$$

$$r_{12} = v_{0(12)}t, \qquad \text{Since, } r_{0(12)} = 0 \text{ and } a_{12} = 0$$

$$\Rightarrow \qquad |r_{12}| = |v_{0(12)}t|$$

But, $|v_{01}| = |v_{02}| = v_0$

$$|v_{0(12)}| = \sqrt{(v_0)^2 + (v_0)^2 - 2v_0v_0\cos\left(\frac{\pi}{2} - \theta_0\right)}$$

Hence, the sought distance is,

$$|r_{12}| = v_0 t\sqrt{2(1 - \sin\theta_0)} = 22 \text{ m}$$

Example 14: The velocity of a projectile when it is at the greatest height is $\sqrt{2/5}$ times its velocity when it is at half of its greatest height. Determine its angle of projection.

Sol: The maximum height is known in terms of initial velocity and angle of projection. Horizontal component of velocity of projectile remains constant. Use the third equation of motion with uniform acceleration in vertical direction to find the vertical component of velocity at height equal to half of the maximum height. Suppose the particle is projected with velocity u at an angle θ with the horizontal. Horizontal component of its velocity at all height will be ucosθ. At the greatest height, the vertical component of velocity is zero, so the resultant velocity is

$$v_1 = u\cos\theta$$

At half the greatest height during upward motion,

$$y = \frac{h}{2}, \quad a_y = -g, \quad u_y = u\sin\theta$$

Using equation of motion,

$$v_y^2 = u^2 \sin^2\theta - (g)\left(\frac{u^2 \sin^2\theta}{2g}\right) = \frac{u^2 \sin^2\theta}{2}$$

$$\Rightarrow \qquad v_y = \frac{u \sin\theta}{\sqrt{2}}$$

Hence, the resultant velocity at half of the greatest height is,

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2 \cos^2\theta + \frac{u^2 \sin^2\theta}{2}}$$

Given, $\frac{v_1}{v_2} = \sqrt{\frac{2}{5}}$

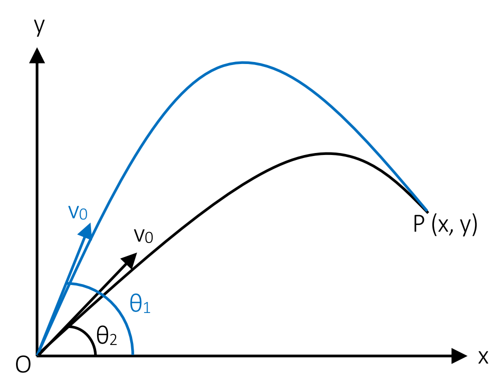
$$\therefore \qquad \frac{v_1^2}{v_2^2} = \left(\frac{u^2 \cos^2\theta}{u^2 \cos^2\theta + \frac{u^2 \sin^2\theta}{2}}\right) = \frac{2}{5}$$

$$\Rightarrow \qquad 2 + \tan^2\theta = 5$$

$$\Rightarrow \qquad \tan\theta = \sqrt{3}$$

$$\Rightarrow \qquad \theta = 60°$$

Example 15: A cannon fires successively two shells with velocity $v_0 = 250\ ms^{-1}$, the first at the angle $\theta_1 = 60°$ and the second at the angle $\theta_2 = 45°$ to the horizontal, the azimuth being the same. Neglecting the air drag, find the time interval between firings leading to the collision of the shells.



Sol: At the instant of collision, the horizontal and vertical distances covered by both the shells is will be equal respectively. Get two equations, one for horizontal distance and the other for vertical distance.

Let the shells collide at the point P(x, y). If the first shell takes t seconds to collide with second and Δt be the time interval between the firings, then,

$$x = v_0 \cos\theta_1 = v_0 \cos\theta_2 (t - \Delta t) \qquad \text{(i)}$$

And $$y = v_0 \sin\theta_2 (t - \Delta t) - \frac{1}{2} g (t - \Delta t)^2 \qquad \text{(ii)}$$

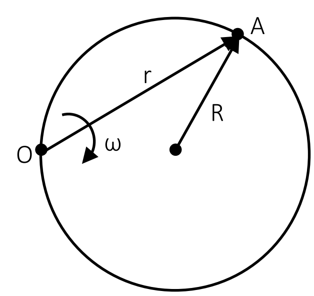
From equation (i),

$$t = \frac{\Delta t \cos\theta_2}{\cos\theta_2 - \cos\theta_1} \qquad \text{(iii)}$$

From equations (ii) and (iii),

$$\Delta t = \frac{2v_0 \sin(\theta_1 - \theta_2)}{g(\cos\theta_2 + \cos\theta_1)} = 11\ s$$

Example 16: A particle A moves along a circle of radius R = 5 cm so that its radius vector r relative to the point O rotates with the constant angular velocity $\omega = 0.40\ rad\ s^{-1}$. Find the modulus of the velocity of the particle, and the modulus and direction of its acceleration.



Sol: Angular velocity about point O is given. We need to find the angular velocity about point C, ω_C. Once ω_C is known, velocity and acceleration can be found out from formulae of circular motion. Angular velocity of point A, with respect to center C of the circle,

$$\omega_C = -\frac{d(2\theta)}{dt} = -2\frac{d\theta}{dt} = 2\omega$$

Because angular speed of line OA is $\omega = -\frac{d\theta}{dt}$

The turning rate of line CA is also the turning rate of velocity vector of point A, which is given by v_A/R.

Therefore, $v_A = \omega_c R = 2\omega R = 4\text{ cm s}^{-1}$ (on substituting the values).

The acceleration of the particle will be centripetal as its speed is constant.

$$a = \frac{v^2}{R} = \frac{4^2}{5} = 3.2\text{ cm s}^{-2}$$

Example 17: A point moves along a circle with a velocity v = kt, where $k = 0.5\text{ ms}^{-2}$. Find the total acceleration of the point at the moment when it has covered the nth fraction of the circle after the beginning of motion,

Where, $n = \frac{1}{10}$

Sol: This is the case of circular motion with constant tangential acceleration. Use second equation of motion with constant acceleration and zero initial velocity to find the time required to cover 1/10 of the circle. Total acceleration is the vector sum of tangential acceleration and centripetal acceleration.

$$v = \frac{ds}{dt} = kt$$

$$\Rightarrow \quad \int_0^s ds = k\int_0^t tdt$$

$$\Rightarrow \quad s = \frac{1}{2}kt^2$$

For completion of nth fraction of the circle,

$$s = (2\pi r)n \text{ or } t^2 = (4\pi nr)/k \qquad \text{(i)}$$

Tangential acceleration, $\alpha_T = \frac{dv}{dt} = k$ (ii)

Normal acceleration , $\alpha_N = \frac{v^2}{r} = \frac{k^2t^2}{r}$ (iii)

$$\Rightarrow \quad \alpha_N = 4\pi nk$$

$$\therefore \quad \alpha = \sqrt{(\alpha_T^2 + \alpha_N^2)} = \sqrt{k^2 + 16\pi^2n^2k^2} = 0.8\text{ ms}^{-2}$$

Example 18: Two boats, A and B move away from a buoy anchored at the middle of a river along mutually perpendicular straight lines: the boat A along the river, and the boat B across the river. Having moved off an equal distance from the buoy the boats returned. Find the ratio of times of motion of boats $\frac{\tau_A}{\tau_B}$ if the velocity of each boat with respect to water is n = 1.2 times greater than the stream velocity.

Sol: The velocity of boat B will be vector sum of velocity of river flow and the velocity of B with respect to river. These three vectors form a right triangle. The velocity of boat B is the base, the velocity of river flow is the perpendicular and the velocity of B with respect to river is the hypotenuse.

Let l be the distance covered by the boat A along the river as well as by the boat B across the river. Let v_0 be the stream velocity and v′ the velocity of each boat with respect to water. Therefore, the time taken by the boat A in its journey,

$$t_A = \frac{l}{v' + v_0} + \frac{l}{v' - v_0} = \frac{2lv'}{v'^2 - v_0^2}$$

And for the boat B,

$$t_B = \frac{l}{\sqrt{v'^2 - v_0^2}} + \frac{l}{\sqrt{v'^2 - v_0^2}} = \frac{2l}{\sqrt{v'^2 - v_0^2}}$$

Hence,

$$\frac{t_A}{t_B} = \frac{v'}{\sqrt{v'^2 - v_0^2}} = \frac{\eta}{\sqrt{\eta^2 - 1}} \qquad \text{where, } \eta = \frac{v'}{v_0}$$

On substitution, $\frac{t_A}{t_B} = 1.8$

Example 19: Two particles move in a uniform gravitational field with an acceleration 'g'. At the initial moment the particles were located at one point and moved with velocities $v_1 = 3.0\ ms^{-1}$ and $v_2 = 4.0\ ms^{-1}$ horizontally in opposite directions. Find the distance between the particles at the moment when their velocity vectors become mutually perpendicular.

Sol: The relative acceleration between the particles is zero. Initial relative distance is zero. So, the final relative distance between them is equal to product of time and relative velocity. The time required can be found by using the equations of final velocities in Cartesian coordinates.

Let the velocities of the particles (say v'_1 and v'_2) become mutually perpendicular after time t. Then, their velocities become,

$$\vec{v}'_1 = v_1\hat{i} + gt\hat{j} \quad \text{and} \quad \vec{v}'_2 = -v_2\hat{i} + gt\hat{j}$$

As, $v'_1 \perp v'_2$, so, $v'_1.v'_2 = 0$

$\Rightarrow$ $(v_1\hat{i} + gt\hat{j}).(-v_2\hat{i} + gt\hat{j}) = 0$

Or, $-v_1v_2 + g^2t^2 = 0$

Hence, $t = \frac{\sqrt{v_1v_2}}{g}$

In the frame attached with 2 for the particle 1,

$$r = r_0 + v_0t + \frac{1}{2}wt^2$$

Both the particles are initially at the same position and have same acceleration g, so,

$$r_0 = 0,\ w = 0 \text{ and } v_0 = |v_1 - v_2|$$

Thus, the sought distance is,

$$|r| = |v_0|t = (v_1 + v_2)t = \frac{(v_1 + v_2)}{g}\sqrt{v_1v_2} = 2.5\ m$$

Example 20: A boy whirls a stone in a horizontal circle of radius 1.5 m and at height 2.0 m above level ground. The string breaks, and the stone flies off horizontally and strikes the ground after travelling a horizontal distance of 10 m. What is the magnitude of the centripetal acceleration of the stone while in circular motion?

Sol: The time of fall of the stone depends on the height of the stone and can be found using the second equation of motion with constant acceleration and zero initial velocity. The horizontal component of stone's velocity remains constant is equal to the horizontal distance covered by the stone divided by the time of fall. The centripetal acceleration is equal to the square of the horizontal velocity divided by the radius of the horizontal circle.

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{4}{9.8}} = 0.64 \text{ s}$$

$$v = \frac{10}{t} = 15.63 \text{ ms}^{-1}$$

$$a = \frac{v^2}{R} = 163 \text{ ms}^{-2}$$

EXERCISE 1 - JEE MAIN (RECTILINEAR AND RELATIVE MOTION)

PART I - SUBJECTIVE QUESTIONS

1. A body starting from rest has an acceleration of 20 ms^{-2}. Calculate the distance travelled by it in 6^{th} second.

2. A train was moving at rate of 36 km h^{-1}. When the brakes were applied, it comes to rest in a distance of 200 m. Calculate the retardation produced in the train.

3. A body covers 12 m in 2^{nd} second and 20 m in 4^{th} second. Find what distance the body will cover in 4 seconds after the 5^{th} second.

4. A racing car moving with constant acceleration covers two successive kilometres in 30 s and 20 s respectively. Find the acceleration of the car.

5. Two cars start off to race with velocities 2 ms^{-1} and 4 ms^{-1} travel in straight line with uniform acceleration 2 ms^{-2} and 1 ms^{-2} respectively. What is the length of the path if they reach the final point at the same time?

6. Brakes are applied to a train travelling at 72 km h^{-1}. After passing over 200 m, its velocity is raced to 36 km h^{-1}. At the same rate of retardation, how much further will it go before it is brought to rest?

7. On turning a corner, a motorist rushing at 44 ms^{-1} finds a child on the road 100 m ahead. He instantly stops the engine and applies the brakes so as to stop it within 1 m of the child. Calculate time required to stop it.

8. A body starting from rest, was observed to cover 20 m in 1 second and 40 m during the next second. How far had it travelled before the first observation was taken?

9. An automobile starts from rest and accelerates uniformly for 30 seconds to a speed of 72 km h^{-1}. It then moves with a uniform velocity and it is finally brought to rest in 50 m with a constant retardation. If the total distance travelled is 950 m, find the acceleration, the retardation and total time taken.

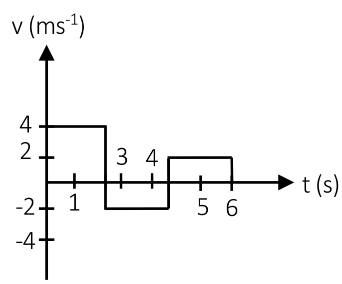
10. From the top of a tower 100 m in height a ball is dropped and at the same instant another ball is projected vertically upwards from the ground so that it just reaches the top of tower. At what height do the two balls pass one another?

11. A body falling from rest was observed to fall through 78.4 m in 2 seconds. Find how long had it been falling before it was observed?

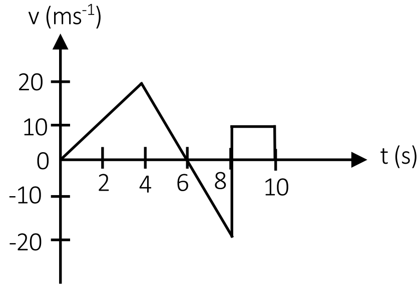
12. A stone is dropped from a balloon at an altitude of 300 m. How long will the stone take to reach the ground if,

(i) The balloon is ascending with a velocity of 5 ms^{-1}.
(ii) The balloon is descending with a velocity of 5 ms^{-1}.
(iii) The balloon is stationary?

13. The velocity-time graph of a body moving in a straight line is shown in the figure. Find the displacement and the distance travelled by the body in 6 seconds.



14. The velocity-time graph of a particle moving along a straight line is as shown in the figure. Calculate the distance covered between t = 0 to t = 10 second. Also find displacement in time 0 to 10 seconds.



15. The position of an object moving along x-axis is given by $x = a + bt^2$, where a = 8.5 m and b = 2.5 ms^{-2} and t is measured in second. What is the velocity at t = 0 s and t = 2.0 s? What is the average velocity between t = 2.0 s and t = 4.01 s?

16. The displacement x (in m) of a body varies with time t (in sec) as $x = -(2/3)t^2 + 16t + 2$. How long does the body take to come to rest?

17. The height y and the distance x along the horizontal, for a body projected in the vertical plane are given by $y = 8t - 5t^2$ and $x = 6t$. What is initial velocity of the body?

18. The displacement of a particle along x-axis is given by $x = 3 + 8t + 7t^2$. Obtain its velocity and acceleration at t = 2s.

19. The relation between time t and distance x is $t = \alpha x^2 + \beta x$ where α and β are constants. Show that retardation is $2\alpha v^3$, where v is the instantaneous velocity.

20. The acceleration 'a' in ms^{-2} of a particle is given by $a = 3t^2 + 2t + 2$, where t is the time. If the particle starts out with a velocity v = 2 ms^{-1} at t = 0, then find the velocity at the end of 2 s.

21. A tennis ball is dropped onto the floor from a height of 4.0 ft. It rebounds to a height of 3.0 ft. If the ball was in contact with the floor for 0.010 s, what was the average acceleration during contact?

22. A particle starts from rest with zero initial acceleration. the acceleration increases uniformly with time. Find the time average and distance average of velocity up to a certain instant when the velocity becomes v.

23. A particle moves along a straight line such that its displacement x from a fixed point on the line at time t is given by $x^2 = at^2 + 2bt + c$ Find acceleration as a function of displacement x.

24. A ball is dropped from a height of 19.6 m above the ground. It rebounds from the ground and raises itself up to the same height. Take the starting point as the origin and vertically downward as the positive x-axis. Draw approximate plots of x versus t, v versus t and a versus t. Neglect the small interval during which the ball was in contact with ground.

25. A train travelling at 72 km h^{-1} is checked by track repairs. It retards uniformly for 200 m covering the next 400 m at constant speed and accelerates uniformly to 72 km h^{-1} in a further 600 m. If the time at constant lower speed is equal to the sum of the times taken in retarding and accelerating. Find the total time taken.

26. A point traversed half the distance with a velocity v_0. The remaining part of the distance was covered with velocity v_1 for half the time, and with velocity v_2 for the other half of the time. Find the men velocity of the point averaged over the whole time of motion.

27. A person sitting on the top of a tall building is dropping balls at regular intervals of one second. Find the positions of the 3rd 4th and 5th ball when the 6th ball is being dropped.

28. A stone is dropped from a balloon going up with a uniform velocity of 5.0 ms^{-1}. if the balloon was 50 m high when the stone was dropped, find its height when the stone hits the ground. [Take g = 10 ms^{-2}]

PART II - OBJECTIVE QUESTIONS

Single Correct Choice Type

1. An object is moving along the x axis with position as a function of time given by x = x(t). Point O is at x = 0. The object is definitely moving towards O when

(A) $dx/dt < 0$ (B) $dx/dt > 0$ (C) $d(x^2)/dt < 0$ (D) $d(x^2)/dt > 0$

2. A particle starts moving rectilinearly at time t = 0 such as that its velocity 'v' changes with time 't' according to the equation $v = t^2 - t$ where t is in seconds and v is in ms^{-1}. The time interval for which the particle retards is

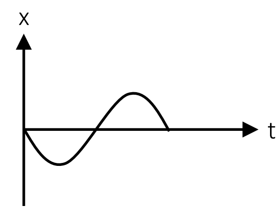
(A) $t < 1/2$ (B) $1/2 < t < 1$ (C) $t > 1$ (D) $t < ½$ and $t > 1$

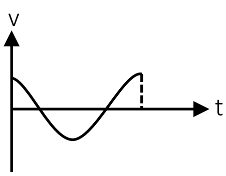
3. An object is tossed vertically into the air with an initial velocity of 8 ms^{-1}. Using the sign convention upwards as positive, how does the vertical component of the acceleration a_y of the object (after leaving the hand) vary during the flight of the object?

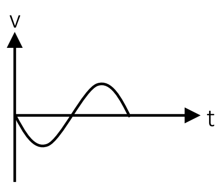
(A) On the way up $a_y > 0$, on the way down $a_y > 0$ (B) On the way up $a_y < 0$, on the way down $a_y < 0$

(C) On the way up $a_y > 0$, on the way down $a_y < 0$ (D) on the way up $a_y < 0$, on the way down $a_y < 0$

4. If position time graph of a particle is since curve as shown, what will be its v-t graph?



(A)


(B)


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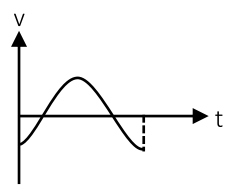
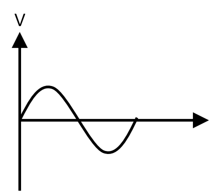


(D)


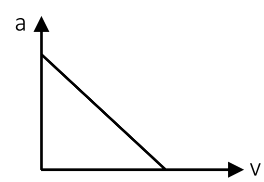
5. A man moves in x-y plane along the path shown. At what point is his average velocity vector in the same direction as his instantaneous velocity vector. The man starts from point P.

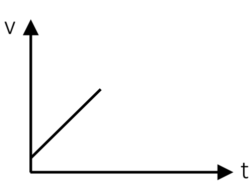
(A) A (B) B (C) C (D) D

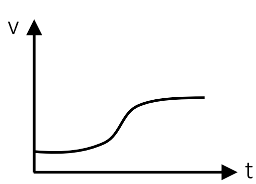
6. The greatest acceleration or deceleration that a train may have is a. The minimum time in which the train reach from one station to the other separated by a distance d is

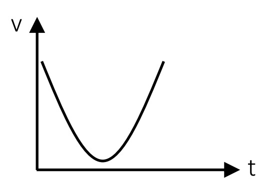
(A) $\sqrt{\frac{d}{a}}$ (B) $\sqrt{\frac{2d}{a}}$ (C) $\frac{1}{2}\sqrt{\frac{d}{a}}$ (D) $2\sqrt{\frac{d}{a}}$

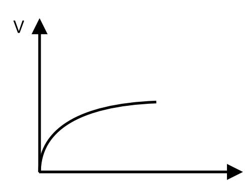
7. Acceleration versus velocity graph of a particle moving in a straight line starting from rest is as shown in figure. The corresponding velocity-time graph would be:



(A)


(B)


(C)


(D)


8. Suppose a player hits several baseballs, which baseball will be in the air for the longest time?

(A) The one with the farthest range.
(B) The one which reaches maximum height.

(C) The one with the greatest initial velocity.
(D) The one leaving the bat at 45° with respect to the ground.

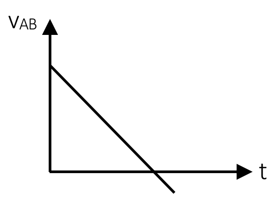
9. A ball is thrown from a point on ground at some angle of projection. At the same time a bird starts from a point directly above this point of projection at a height h horizontally with speed u. Given that in its flight ball just touches the bird at one point. Find the distance on ground where ball strikes.

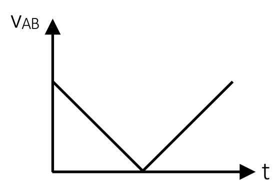
(A) $2u\sqrt{\frac{h}{g}}$ (B) $u\sqrt{\frac{2h}{g}}$ (C) $2u\sqrt{\frac{2h}{g}}$ (D) $u\sqrt{\frac{h}{g}}$

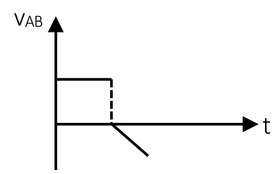
10. It takes one minute for a passenger standing on an escalator to reach the top. If the escalator does not move it takes him 3 minutes to walk up. How long will it take for the passenger to arrive at the top if he walks up the moving escalator?

(A) 30 sec (B) 45 sec (C) 40 sec (D) 35 sec

11. A body A is thrown vertically upwards with such a velocity that it reaches a maximum height of h. simultaneously another body B is dropped from height h. It strikes the ground and does not rebound. The velocity of A relative to B v/s time graph is best represented by: (upward direction is positive)

(A)


(B)


(C)


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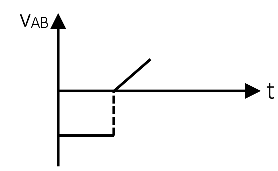
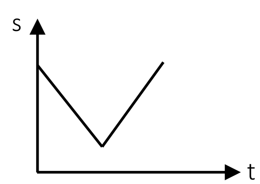
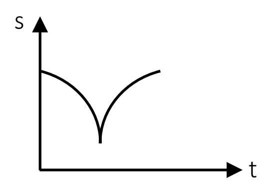
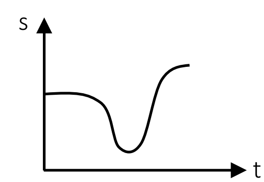
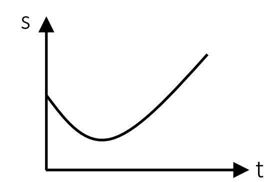


12. Particle A and B are moving with constant velocities along x and y axis respectively, the graph of separation between them with time is

(A)


(B)


(C)


(D)


EXERCISE 2 - JEE ADVANCED

PART I - SUBJECTIVE QUESTIONS

1. A car moving along a straight highway with a speed of 126 km h^{-1} is brought to stop with in a distance of 200 m. what is the retardation of the car (assumed uniform) and how long does it take for the car to stop?

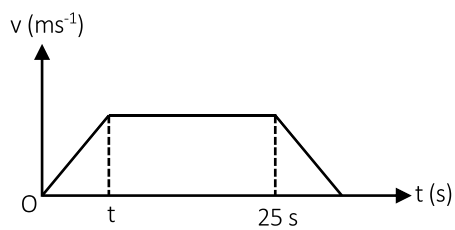
2. A police van moving on a highway with a speed of 30 km h^{-1} fires a bullet at a thief's car speeding away in the same direction with a speed of 192 km h^{-1}, if the muzzle speed of bullet is 150 ms^{-1}, with what speed does the bullet hit the thief's car.

3. What is the ratio of the distance travelled by a body falling freely from rest during first, second and third second of its fall.

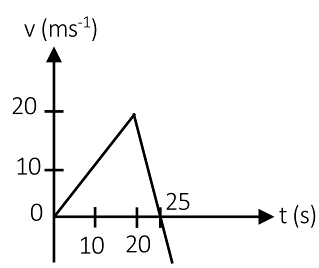
4. At a distance L = 400 m from the traffic light breaks are applied to locomotive moving at a velocity v = 54 km h^{-1}. Determine the position of the locomotive relative to the traffic light 1 minute after the application of the breaks if its acceleration is -0.3 ms^{-2}.

5. An object moving with uniform acceleration has a velocity of 12 cm s^{-1} in the positive x direction when its x coordinate is 3.00 cm. If its x coordinate 2.00 s later is - 5.00 cm, what is its acceleration?

6. The velocity-time graph of the particle moving along the straight line is shown. The rate of acceleration and deceleration is constant and it is equal to 5 ms^{-2}. If the average velocity during the motion is 20 ms^{-2}, then the final value of t.



7. The figure shows the v-t graph of a particle moving in the straight line. Find the time when particle returns to the starting point.



8. A stone is dropped from a height h. simultaneously another stone is thrown up from the ground with such a velocity that it can reach a height of 4h. Find the time when two stones cross each other.

9. A glass wind screen whose inclination with the vertical can be changed is mounted on a car. The car moves horizontally with a speed of 2 ms^{-1}. At what angle α with the vertical should the wind screen be placed so that the rain drops falling vertically downwards with velocity 6 ms^{-1} strikes the wind screen perpendicularly?

10. Two particles are moving along two straight lines, in the same plane, with the same speed = 20 cm s^{-1}. The angle between the two lines is 60°, and their intersection point is O. At a certain moment, the two particles are located at distances 3 m and 4 m from O, and are moving towards O. Find the shortest distance between them subsequently?

11. A point mass starts moving in a straight line with a constant acceleration a. At a time t, after the beginning of motion, the acceleration changes sign, remaining the same magnitude. Determine the time t from the beginning of motion in which the point mass returns to the initial position?

12. For $\left(\frac{1}{m}\right)^{th}$ of the distance between two stations, a train is uniformly accelerated and for $\left(\frac{1}{n}\right)^{th}$ of the distance, it is uniformly retarded. It starts from rest at one station and comes to rest at another. Find the ratio of its maximum velocity to its average velocity?

13. The velocity of a particle moving in the positive direction of the x axis varies as $v = \alpha\sqrt{x}$, where α is positive constant. Assuming that at the moment t = 0 the particle was located at the point x = 0, find:

(i) The time dependence of the velocity and acceleration of the particle.
(ii) The mean velocity of the particle averaged over the time that the particle takes to cover the first s meter of the path.

14. A boy standing on a long railroad car throws a ball straight upwards. The car is moving on the horizontal road with an acceleration of 1 ms^{-2} and projection velocity in the vertical direction is 9.8 ms^{-1}. How far behind the boy will the ball fall on the car?

15. A person walks up a stalled escalator in 90 s. When standing on the same escalator, now moving, he is carried up to 60 s. How much time would it take him to walk up the moving escalator?

16. Two trains of lengths 180 m are moving on parallel tracks. If they move in the same direction then they cross each other in 15 s, and if they move in opposite directions then they cross in 7.5 seconds, then calculate their velocities.

17. At the instant the traffic light turns green, an automobile starts with a constant acceleration a_x of 6.0 ft s^{-2} , at the same instant a truck, travelling with a constant speed of 30 ft s^{-2}, overtakes and passes the automobile.

(i) How far beyond the straight point will automobile overtake the truck?
(ii) How fast will the automobile be travelling at that instant? (It is instructive to plot qualitative graph of x versus t for each vehicle.)

18. Two bodies moves in a straight line towards each other at initial velocities v_1 and v_2 and constant accelerations a_1 and a_2 directed against the corresponding velocities at the initial instant. What must be the maximum initial separation l_{max} between the bodies for which they meet during motion?

19. An ant runs from an ant-hill in a straight line so that its velocity is inversely proportional to the distance from the centre of the ant-hill. When the ant is at point A at a distance l_1 = 1 m from the centre of the ant-hill, its velocity v_1 = 2 cm s^{-1} , what time will it take ant to run from point A to point B, which is at a distance l_2 = 2 m from the centre of the ant-hill?

20. Distance between two points A and B is 33 m. A particle P starts from B with velocity of 1 ms^{-1} along AB with an acceleration of 2 ms^{-2}. Simultaneously another particle Q starts from A with a velocity of 9 ms^{-1} in the same direction AB and has an acceleration 1 ms^{-2} in the direction AB. Find whether Q will be able to catch P.

PART II - OBJECTIVE QUESTIONS

Multiple Correct Choice Type

1. A particle moves with constant speed v along a regular hexagon ABCDEF in the same order. Then the magnitude of the average velocity for its motion from A to

(A) F is v/5 (B) D is v/3 (C) C is $v\sqrt{3}/2$ (D) B is v

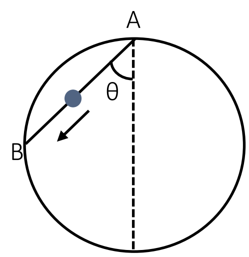
2. A particle moving with a speed v changes direction by an angle θ , without change in speed.

(A) The change in the magnitude of its velocity is zero.
(B) The change in the magnitude of its velocity is 2vsin(θ/2).
(C) The magnitude of change in velocity is 2vsin(θ/2).
(D) The magnitude of change in velocity is 2v(1 - cos θ).

3. A particle has initial velocity 10 ms^{-1}. It moves due to constant retarding force along the line of velocity which produces a retardation of 5 ms^{-2}. Then

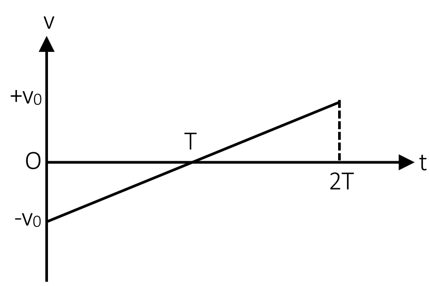
(A) the maximum displacement in the direction of initial velocity is 10 m.
(B) the distance travelled in first 3 seconds is 7.5 m.
(C) the distance travelled in the first 3 seconds is 12.5 m.
(D) the distance travelled in the first 3 seconds is 17.5 m.

4. A bead is free to slide down a smooth wire tightly stretched between points A and B on a vertical circle. If the bead starts from rest at A, the highest point on the circle



(A) Its velocity v on arriving at B is proportional to cos θ
(B) Its velocity v on arriving at B is proportional to tan θ
(C) Time to arrive at B is proportional to cos θ
(D) Time to arrive at B is independent of θ .

5. The figure shows the velocity (v) of a particle plotted against time (t)



(A) The particle changes its direction of motion at some point.
(B) The acceleration of the particle remains constant.
(C) The displacement of the particle is zero.
(D) The initial and final speed of the particle are the same.

6. An observer moves with a constant speed along the line joining two stationary objects. He will observe that the two objects

(A) Have the same speed
(B) Have the same velocity
(C) Move in the same direction
(D) Move in opposite directions

7. A man on a rectilinearly moving cart, facing the direction of motion, throws a ball straight up with respect to himself

(A) The ball will always return to him.
(B) The ball will never return to him.
(C) The ball will return to him if the cart moves with constant velocity.
(D) The ball will fall behind him if the cart moves with some positive acceleration.

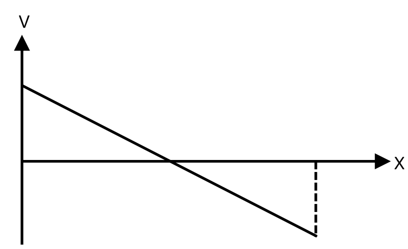
Assertion Reasoning Type

(A) Statement-I is true, statement-II, is true and statement-II is correct explanation for statement-I
(B) Statement-I is true, statement-II is true and statement-II is NOT the correct explanation for statement-I.
(C) Statement-I is true, statement-II is false.
(D) Statement-I is false, statement-II is true.

8. **Statement-I:** Positive acceleration in rectilinear motion of a body does not imply that the body is speeding up
Statement-II: Both the acceleration and velocity are vectors.

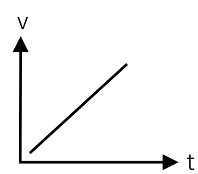
9. **Statement-I:** A particle having zero acceleration must have constant speed.
Statement-II: A particle having zero acceleration must have zero acceleration.

10. **Statement-I:** A student performed an experiment by moving a certain block in a straight line. The velocity position graph cannot be as shown.



Statement-II: When a particle is at its maximum displacement in rectilinear motion its velocity must be zero.

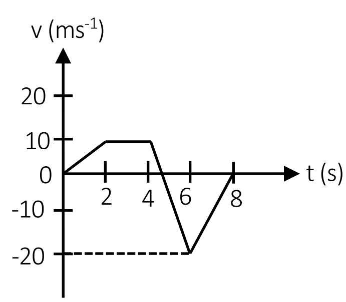
11. **Statement-I:** If the velocity time graph of a body moving in a straight line is as shown here, the acceleration of the body must be constant.



Statement-II: The rate of change of quantity which is constant is always zero.

Comprehension Type

Paragraph 1: The figure shows a velocity-time graph of a particle moving along a straight line



12. Choose the incorrect statement. The particle comes to rest at

(A) t = 0 s (B) t = 5 s (C) t = 8 s (D) None of these

13. Identify the region in which the rate of change of velocity $\left|\frac{\Delta \vec{v}}{\Delta t}\right|$ of the particle is maximum

(A) 0 to 2 s (B) 2 to 4 s (C) 4 to 6 s (D) 6 to 8 s

14. If the particle starts from the position x_0 = - 15 m, then its position at t = 2 s will be

(A) - 5 m (B) 5 m (C) 10 m (D) 15 m

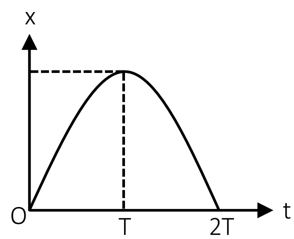
15. The maximum displacement of the particle is

(A) 33.3 m (B) 23.3 m (C) 18.3 m (D) Zero

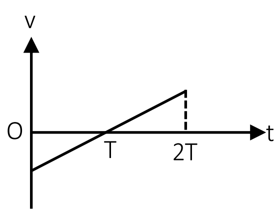
16. The total distance travelled by the particle is

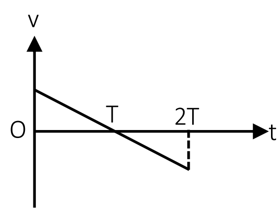
(A) 66.7 m (B) 51.6 m (C) Zero (D) 36.6 m

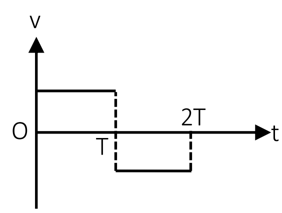
Paragraph 2: The x-t graph of the particle moving along a straight line is shown in the figure

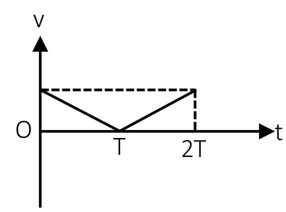


17. The v-t graph of the particle is correctly shown by

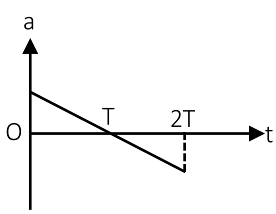
(A)


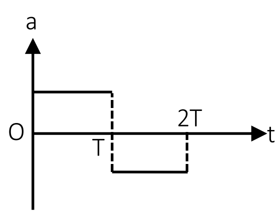
(B)


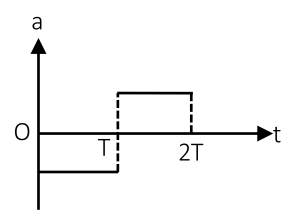
(C)


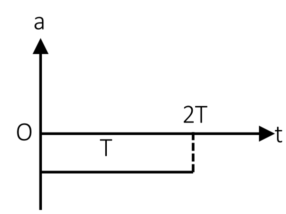
(D)


18. The a-t graph of the particle is correctly shown by

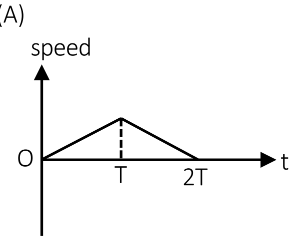
(A)


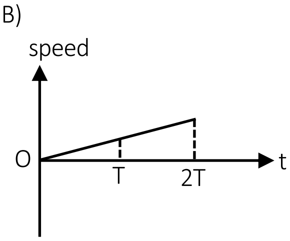
(B)


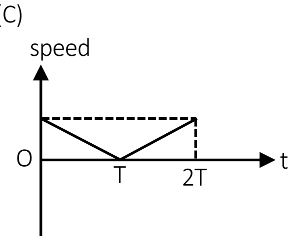
(C)


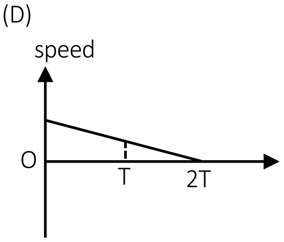
(D)


19. The speed–time graph of the particle is correctly shown by

(A)


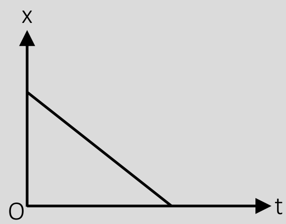
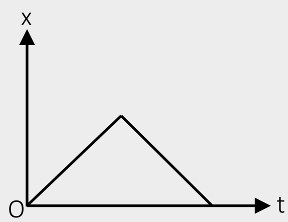
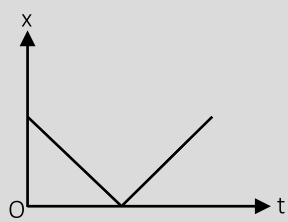
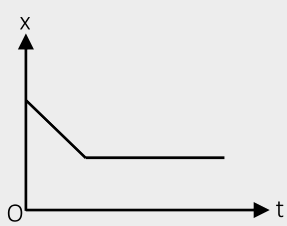
(B)


(C)


(D)


Match the Columns

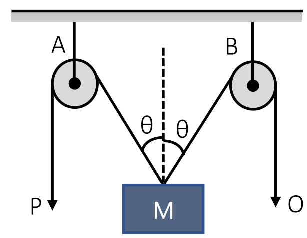
20. Column I shows position versus time graph for an object and column II shows possible graphs.

Column I	Column II
(A) 	(p) A ball rolls along the floor towards the origin
(B) 	(q) A ball rolled towards a wall at the origin, then ball rebounds.
(C) 	(r) A ball rolling away from the origin; hits a wall and bounces straight back.
(D) 	(s) An object rolling towards the origin and suddenly stops.
	(t) A book at rest on a table.

EXERCISE 3 - PREVIOUS YEARS' PROBLEMS

PART I - JEE MAIN

1. In the arrangement shown in the Figure. the ends P and Q of an unstretchable string move downwards with uniform speed U. Pulleys A and B are fixed. Mass M moves upwards with s speed **[AIEEE 1982]**



(A) $2U\cos\theta$ (B) $U/\cos\theta$ (C) $2U/\cos\theta$ (D) $U\cos\theta$

2. A particle is moving eastwards with a velocity of 5 ms^{-1}. In 10 s the velocity changes to 5 ms^{-1} northwards. The average acceleration in this time is **[AIEEE 1982]**

(A) Zero

(B) $\frac{1}{\sqrt{2}}$ ms^{-2} towards north-east

(C) $\frac{1}{\sqrt{2}}$ ms^{-2} towards north-west

(D) $\frac{1}{2}$ ms^{-2} towards north

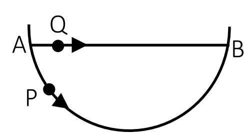
3. A river is flowing from west to east at a speed of 5 m min^{-1}. A man on the south bank of the river, capable of swimming at 10 m min^{-1} in still water to swim across the river in the shortest time. He should swim in a direction. **[AIEEE 1983]**

(A) Due north (B) 30° east of north (C) 30° west of north (D) 60° east of north

4. A boat which has a speed of 5 km h^{-1} still crosses a river of width 1 km along the shortest possible path in 15 min. The velocity of the river water in km h^{-1} is **[AIEEE 1988]**

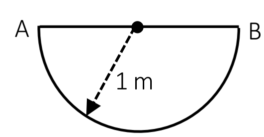
(A) 1 (B) 3 (C) 4 (D) $\sqrt{41}$

5. A particle P is sliding down a frictionless hemispherical bowl. It passes the point A at t = 0. At this instant of time, the horizontal component of its velocity is v. A bead Q of the same mass as P is ejected from A at t = 0 along the horizontal string mass AB, with the speed v. Friction between the bead and the string may be neglected. Let t_P and t_Q be the respective times taken by P and Q to reach the point B. Then **[AIEEE 1993]**



(A) $t_P < t_Q$ (B) $t_P = t_Q$ (C) $t_P > t_Q$ (D) $\frac{t_P}{t_Q} = \frac{\text{lengthof arcACB}}{\text{lengthof chord AB}}$

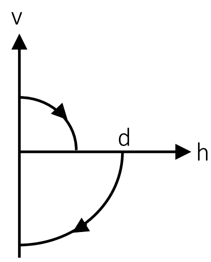
6. In 1.0 s, a particle goes from point A to point B, moving in a semicircle (see the Figure). The magnitude of the average velocity is **[AIEEE 1999]**



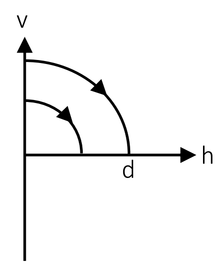
(A) 3.14 ms^{-1} (B) 2.0 ms^{-1} (C) 1.0 ms^{-1} (D) Zero

7. A ball is dropped vertically from a height d above the ground. It hits the ground and bounces up vertically to a height d/2. Neglecting subsequent motion and air resistance, its velocity v varies with height h above the ground as **[AIEEE 2000]**

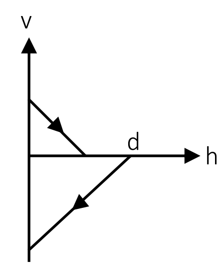
(A)



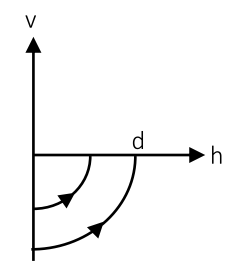
(B)



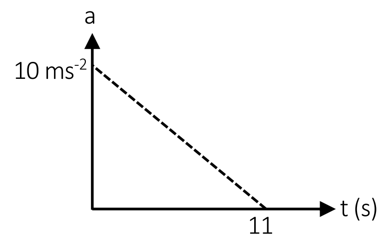
(C)



(D)



8. A particle starts from rest. Its acceleration (a) versus time (t) is as shown in the Figure. The maximum speed. The maximum speed of the particle will be **[AIEEE 2004]**



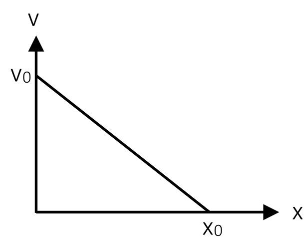
(A) 110 ms^{-1} (B) 55 ms^{-1} (C) 550 ms^{-1} (D) 660 ms^{-1}

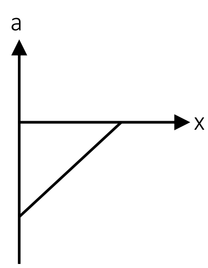
9. A small block slides without friction down an inclined plane starting from rest. Let n s be the distance travelled from t = n - 1 to t = n. **[AIEEE 2004]**

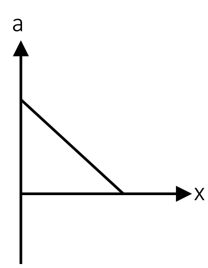
Then, $\frac{s_n}{s_n + 1}$ is

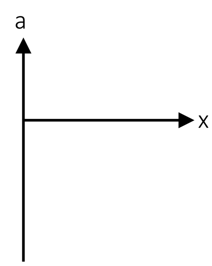
(A) $\frac{2n - 1}{2n}$ (B) $\frac{2n + 1}{2n - 1}$ (C) $\frac{2n - 1}{2n + 1}$ (D) $\frac{2n}{2n + 1}$

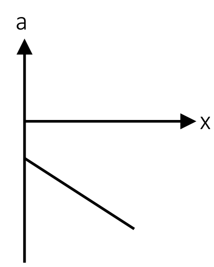
10. The given graph shows the variation of velocity with displacement. Which of the graph given below correctly represents the variation of acceleration with displacement? **[AIEEE 2005]**



(A)


(B)


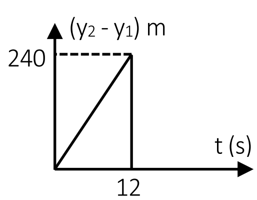
(C)


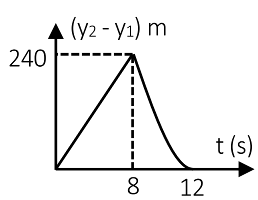
(D)


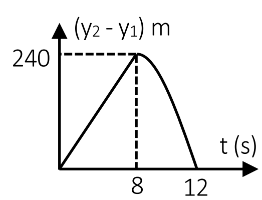
11. From a tower of height H, a particle is thrown vertically upwards with a speed u. The time taken by the particle, to hit the ground, is n times that taken by it to reach the highest point of its path. The relation between H, u and n is: **[JEE MAIN 2014]**

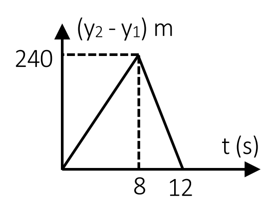
(A) $2gH = nu^2(n - 2)$ (B) $gH = (n - 2)u^2$ (C) $2gH = n^2u^2$ (D) $gH = (n - 2)^2u^2$

12. Two stones are thrown up simultaneously from the edge of a cliff 240 m high with initial speed of 10 ms^{-1} and 40 ms^{-1} respectively. Which of the following graph best represents the time variation of relative position of the second stone with respect to the first? (Assume stones do not rebound after hitting the ground and neglect air resistance, take g = 10 ms^{-2})(The figures are schematic and not drawn to scale) **[JEE MAIN 2015]**

(A)


(B)


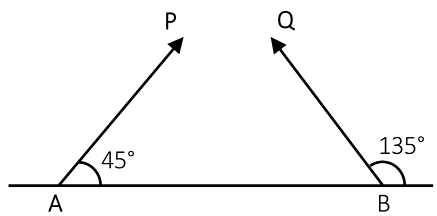
(C)


(D)


PART II - JEE ADVANCED

1. Particles P and Q of mass 20 g and 40 g respectively are simultaneously projected from points A and B on the ground. The initial velocities of P and Q makes 45° and 135° angles respectively with the horizontal AB as shown in the Fig. Each particle has an initial speed of 49 ms^{-1}. The separation AB is 245 m.

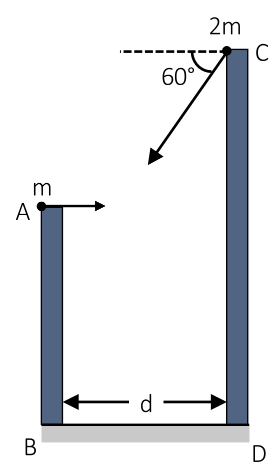
Both particles travel in the same vertical plane and undergo a collision. After the collision, P retraces its path. Determine the position Q where it hits the ground. How much time after collision does the particle Q take to reach the ground? [Take g = 9.8 ms^{-2}] **[IIT JEE 1982]**



2. A body falling freely from a given height H hits an inclined plane in its path at a height h. As a result of this impact the direction of the velocity of the body becomes horizontal. For what value of (h/H) the body will take maximum time to reach the ground? **[IIT JEE 1986]**

3. Two towers AB and CD are situated a distance d apart as shown in Figure. AB is 20 m high and CD is 30 m high from the ground. An object of mass m is thrown from the top of AB horizontally with a velocity of 10 ms^{-1} towards CD. Simultaneously other object of mass 2 m is thrown from top of CD at an angle of 60° to the horizontal towards AB with the same magnitude of initial velocity as that of the first object. The two objects moves in the same vertical plane, collide in mid-air and stick to each other. **[IIT JEE 1994]**

(i) Calculate the distance d between the towers.
(ii) Find the position where the objects hit the ground.



4. Two guns situated on the top of a hill of height 10 m fire one shot each with same speed $5\sqrt{3}$ ms^{-1} at some interval of time. One gun fires horizontally and other fires upwards at an angle of 60° with the horizontal. The shots collide in air at point P. Find:

(i) The time interval between the firings
(ii) The coordinates of the point P. Take origin of the coordinate system at the foot of the hill right below the muzzle and trajectories in x-y plane. [Take g = 10 ms^{-2}] **[IIT JEE 1996]**

5. A cart is moving along x-direction with a velocity of 4 ms^{-1}. A person on the cart throws a stone with a velocity of 6 ms^{-1} relative to himself. In the frame of reference of the cart the stone is thrown in y-z plane making an angle of 30° with vertical z-axis. At the highest point of its trajectory the stone hits an object of equal mass hung vertically from the branch of a tree by means of a string of length L. A completely inelastic collision occurs, in which the stone gets embedded in the object. Determine: **[IIT JEE 1997]**

(i) The speed of the combined mass immediately after the collision with respect to an observer on the ground.
(ii) The length L of the string such that the tension in the string becomes zero when the string becomes horizontal during the subsequent motion of the combined mass. [Take g = 9.8 ms^{-2}]

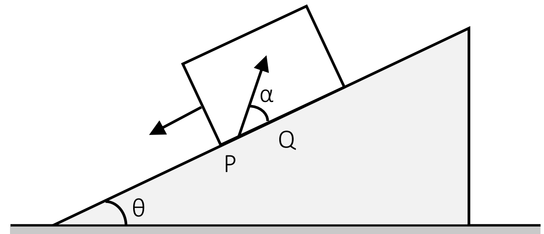
6. A particle of mass 10^{-2} kg is moving along the positive x-axis under the influence of a force $F(x) = -k/2x^2$ where $k = 10^{-2}$ Nm^2. At time t = 0 it is at x = 1.0 m and its velocity v = 0.

(i) Find its velocity when it reaches x = 0.5 m.
(ii) Find the time at which it reaches x = 0.25 m. **[IIT JEE 1998]**

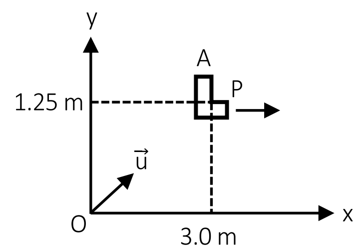
7. A large heavy box is sliding without friction down a smooth plane of inclination θ. From a point P on the bottom of the box, a particle is projected inside the box. The initial speed of the particle with respect to box is u and the direction of projection makes an angle α with the bottom as shown in the figure. **[IIT JEE 1998]**

(i) Find the distance along the bottom of the box between the point of projection P and point Q where the particle lands (Assume that the particle does not hit any other surface of the box. Neglect air resistance.)

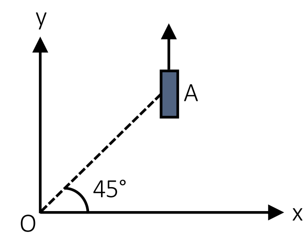
(ii) If the horizontal displacement of the particle as seen by an observer on the ground is zero, find the speed of the box with respect to the ground at the instant when the particle was projected.



8. An object A is kept fixed at the point x = 3 and y = 1.25 m on a plank P raised above the ground. At time t = 0 the plank starts moving, along the + x-direction with an acceleration 1.5 ms^{-2}. At the same instant a stone is projected from the origin with velocity $\vec{u}$ as shown. A stationery person on the ground observes the stone hitting the object during its downward motion at an angle of 45° to the horizontal. All the motions are in x-y plane, Find $\vec{u}$ and the time after which the stone hits the object. [Take g = 10 ms^{-2}]**[IIT JEE 2000]**



9. On a frictionless horizontal surface, assumed to be the x-y plane, a small trolley A is moving along a straight line parallel to the y-axis with a constant velocity of $(\sqrt{3} - 1)$ ms^{-1}. At a particular instant when the line OA makes an angle of 45° with the x-axis, a ball is thrown along the surface from origin O. Its velocity makes an angle ϕ with the x-axis and it hits the trolley. **[IIT JEE 2002]**



10. A train is moving along a straight line with a constant acceleration a. A boy standing in the train throws a ball forward with a speed of 10 ms^{-1}, at an angle of 60° to the horizontal. The boy has to move forward by 1.15 m inside the train to catch the ball back at the initial height. The acceleration of the train in ms^{-2} is: **[IIT JEE 2011]**

Assertion Reasoning Type

Mark your answer as

(A) If Statement-I is true, statement-II is true: statement-II is the correct explanation for statement-I.
(B) If Statement-I is true, statement-II is true: statement-II is not a correct explanation for statement-I.
(C) If Statement-I is true: statement-II is false.
(D) If Statement-I is false: statement-II is true.

11. **Statement-I:** For an observer looking out through the window of a fast moving train, the nearby objects appear to move in the opposite direction to the train, while the distant objects appear to be stationary.

Statement-II: If the observer and the object are moving at velocities $\vec{v}_1$ and $\vec{v}_2$ respectively with reference to a laboratory frame, the velocity of the object with respect to the observer is $\vec{v}_2 - \vec{v}_1$ **[IIT JEE 2008]**

12. A particle of mass m moves on the x-axis as follows: it starts from rest at t = 0 from the point x = 0, and comes to rest at t = 1 at the point x = 1. No other information is available about its motion at intermediate times (0 < t < 1). If α denotes the instantaneous acceleration of the particle, then **[IIT JEE 1993]**

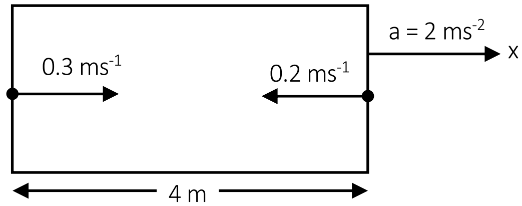
(A) α cannot remain positive for all t in the interval 0 ≤ t ≤ 1
(B) |α| cannot exceeds 2 at any point in its path
(C) |α| must be ≥ 4 at some point or points in its path
(D) α must change sign during the motion, but no other assertion can be made with information given.

13. The coordinates of a particle moving in a plane are given by x(t) = acos(pt) and y(t) = bsin(pt) where a,b (b < a) and p are positive constants of appropriate dimensions. Then

(A) the path of the particle is an ellipse
(B) the velocity and acceleration of the particle are normal to each other at t = π/2p

(C) the acceleration of the particle is always directed towards a focus.
(D) the distance travelled by the particle in time interval t = 0 to t = π/2p is a.

14. A rocket is moving in a gravity free space with a constant acceleration of 2 ms^{-2} along + x direction (see figure). The length of a chamber inside the rocket is 4 m. A ball is thrown from the left end of the chamber in + x direction with a speed of 0.3 ms^{-1} relative to the rocket. At the same time, another ball is thrown in - x direction with a speed of 0.2 ms^{-1} from its right end relative to the rocket. The time in seconds when the two balls hit each other is **[JEE ADVANCED 2014]**



ANSWERS

EXERCISE 1

PART I

1. 110 m
2. 0.25 ms^{-2}
3. 136 m
4. 2/3 ms^{-2}
5. 24 m
6. 66.67 m
7. 4.5 s
8. 2.5 m
9. 2/3 ms^{-2}, 4 ms^{-2}, 65 s
10. 75 m from ground
11. 3 s
12. (i) 8.36 s (ii) 7.33 s (iii) 7.82 s
13. 8 m, 16 m
14. 100 m, 60 m
15. 0 ms^{-1}; 10 ms^{-1}; 15 ms^{-1}
16. 12 s
17. 10 ms^{-2}
18. 36 ms^{-1}; 14 ms^{-1}
20. 18 ms^{-1}
21. Approximately, 3000 ft s^{-2} (in the upward direction)
22. $< v >_{time} = 3v/5$
23. $\frac{ac - b^2}{x^3}$
25. 2 min
26. $\frac{2v_0(v_1 + v_2)}{v_1 + v_2 + 2v_0}$
27. 45 m, 20 m, 5 m
28. 68.5 m

PART II

1. C **2.** B **3.** D **4.** C **5.** C **6.** D **7.** D **8.** B **9.** C **10.** B **11.** C **12.** D

EXERCISE 2

PART I

1. 11.43 s, - 3.06 ms^{-2}
2. 105 ms^{-2}
3. 1:3:5
4. 25 m
5. - 16 cm s^{-2}
6. 5 s
7. 36.2 s
8. $\sqrt{h/8g}$
9. $\tan^{-1}(3)$
10. $50\sqrt{3}$ cm
11. $t = t_1(2 + \sqrt{2})$
12. $\left[1 + \frac{1}{m} + \frac{1}{n}\right]:1$
13. (i) $v = \frac{\alpha^2 t}{2}$, $a = \frac{\alpha^2}{2}$ (ii) $\vec{v}$ $\frac{\alpha\sqrt{s}}{2}$
14. 2 m
15. 36 s
16. 36 ms^{-1}, 12 ms^{-1}
17. (i) 300 ft (ii) 60 ft s^{-1}
18. $l_{max} = \frac{(v_1 + v_2)}{2(a_1 + a_2)}$
19. t = 75 s
20. Q cannot catch P

PART II

1. A, C, D **2.** A, C **3.** A, C **4.** A, D **5.** A, B, C, D **6.** A, B, C

7. C, D	**8.** A	**9.** D	**10.** A	**11.** B	**12.** B
13. C	**14.** A	**15.** A	**16.** A	**17.** B	**18.** D
19. C	**20.** A → p, s; B → r ; C → q ; D → s				

EXERCISE 3

PART I

1. B **2.** C **3.** A **4.** B **5.** A **6.** B **7.** A **8.** B **9.** C **10.** A **11.** A **12.** B

PART II

1. Just midway between A and B, 3.53 s

2. 1/2

3. (i) Approximately 17.32 m (ii) 11.55 m from B

4. (i) 1 s (ii) $(5\sqrt{3}$ m, 5 m)

5. (i) 2.5 ms^{-1} (ii) 0.32 m

6. (i) - 0.1 ms^{-1} (ii) 1.48 s

7. (i) $\frac{u^2 \sin 2\alpha}{g \cos \theta}$ (ii) $\frac{u \cos(\alpha + \theta)}{\cos \theta}$ (down the plane)

8. $\vec{u} = \left(3.75\hat{i} + 6.25\hat{j}\right)$ ms^{-1}, 1 s

9. (i) 45° (ii) 2 ms^{-1}

10. 5

11. B

12. A, C

13. A, B, C

EXERCISE 1 - JEE MAIN (PROJECTILE AND CIRCULAR MOTION)

PART I - SUBJECTIVE QUESTIONS

Projectile Motion

1. What do you understand by motion in two dimensions? When an object is moving with uniform velocity in two dimensions, explain displacement, velocity and find the equations of motion of the object.

2. Find the relation for (i) velocity and time (ii) displacement and time, when an object is moving with uniform acceleration in two dimensions.

3. What is a projectile? Give its examples. Show that the path of projectile is a parabolic path when projected horizontally from a certain height.

4. Show that there are two angles of projection for which the horizontal range is the same.

5. Find (i) time of flight, (ii) Max. height and (iii) horizontal range of projectile projected with speed

v_{AB} (Acceleration of A with respect to B) = $v_A - v_B$

a_{AB} (Acceleration of A with respect to B) = $a_A - a_B$

making an angle θ with the horizontal direction from ground.

6. Find the magnitude and direction of the velocity of an object at any instant during the oblique projection of a projectile.

7. Find (i) the path of projectile, (ii) time of flight, (iii) horizontal range and (iv) maximum height, when a projectile is projected with velocity υ making an angle θ with the vertical direction.

8. What is centripetal acceleration? Find its magnitude and direction in case of a UCM of an object.

9. A stone is dropped from the window of a bus moving at 60 km h^{-1}. If the window is 196 cm high, find the distance along the track which the stone moves before striking the ground.

10. A hiker stands on the edge of a cliff 490 m above the ground and throws a stone horizontally with an initial speed of 15 ms^{-1}. Neglecting air resistance, find the time taken by the stone to reach the ground, and the speed with which it hits the ground. [Take $g = 9.8$ ms^{-2}]

11. A ball is thrown horizontally from the top of a tower with a speed of 50 ms^{-1}. Find the velocity and position at the end of 3 second. [Take $g = 9.8$ ms^{-2}]

12. A body is projected downward at an angle of 30° to the horizontal with a velocity of 9.8 ms^{-1} from the top of a tower 29.4 m high. How long will it take before striking the ground?

13. Prove that a gun will shoot three times as high when its angle of elevation is 60° as when it is 30° , but cover the same horizontal range.

14. Prove that the maximum horizontal range is 4 times the maximum height attained by a projectile which is fired along the required oblique direction.

15. Two particles are projected from the ground simultaneously with speeds of 30 ms^{-1} and 20 ms^{-1} at angles 60° and 30° with the horizontal on the same direction. Find maximum distance between them on ground where they strike. [Take $g = 10$ ms^{-2}]

16. A projectile has the same range when the maximum height attained by it is either H_1 or H_2 . Find the relation between R, H_1 and H_2.

17. A projectile is given an initial velocity of $(\hat{i} + 2\hat{j})$, where $\hat{i}$ is a unit vector along horizontal and $\hat{j}$ is unit vector vertically upward. Find the Cartesian equation of its path. [Take $g = 10$ ms^{-2}]

18. Find the maximum horizontal range of a cricket ball projected with a velocity of 80 ms^{-1}. If the ball is to have a range of $100\sqrt{3}$ m, find the least angle of projection and the least time taken.

19. A bullet fired from a rifle attains a maximum height of 5 m and crosses a range of 200 m. Find the angle of projection.

20. A target is fixed on the top of a pole 13 m high. A person standing at a distance 50 m from the pole is capable of projecting a stone with a velocity $10\sqrt{g}$ ms^{-1}. If he wants to strike the target in shortest possible time, at what angle should he project the stone.

21. A particle is projected with a velocity u so that its horizontal range is twice the greatest height attained. Find the horizontal range of it.

22. A drunkard walking in a narrow lane takes 5 steps forward and 3 steps backward, and so on. Each step is 1 m long and required 1 s. Determine how long the drunkard takes to fall in a pit 13 m away from the start.

23. A jet airplane travelling at the speed of 500 km h^{-1} ejects its projects of combustion at the speed of 1500 km h^{-1} relative to the jet plane. What is the speed of the later with respect to observer on the ground?

24. A car moving along a straight highway with speed of 126 km h^{-1} is brought to a stop within a distance of 200 m. What is the retardation of the car (assumed uniform) and how long does it take for the car to stop?

25. Two trains A and B of length 400 m each are moving on two parallel tracks with a uniform speed of 72 km h^{-1} in the same direction with A head of B. The driver of B decides to overtake A and accelerate by 1 ms^{-2} . If after 50 s, the guard of B just brushes past the driver of A, what was the original distance between them?

26. Two towns A and B are connected by a regular bus service with a bus leaving in either direction every T min. A man cycling with a speed of 20 km h^{-1} in the direction of A and B notices that a bus goes past him every 18 min in the direction of his motion, and every 6 min in the opposite direction. What is the period T of the bus service and with what speed (assumed constant) do the buses ply on the road?

27. A ball is dropped from a height of 90 m on a floor. At each collision with the floor, the ball loses one tenth of its speed. Plot the speed-time graph of its motion between t = 0 to 12 s. [Take g = 10 ms^{-2}]

28. A man walks on a straight road from his home to a market 2.5 km away with a speed of 5 km h^{-1}. Finding the market closed, he instantly turns and walks back with a speed of 7.5 km h^{-1}. What is the (a) magnitude of average velocity and (b) average speed of the man, over the interval of time (i) 0 to 30 min. (ii) 0 to 50 min (iii) 0 to 40 min?

29. A dive bomber, diving at an angle of 53° with the vertical, released a bomb at an altitude of 2400 ft. The bomb hits the ground 5.0 s after being released. (i) What is the speed of the bomber? (ii) How far did the bomb travel horizontally during its flight? (iii) What were the horizontal and vertical components of its velocity just before striking the ground?

Circular Motion

30. Calculate the angular velocity of the minute's hand of a clock.

31. What is the angular velocity in radian per second of a fly wheel making 300 r.p.m.?

32. The wheel of an automobile is rotating with 4 rotations per second. Find its angular velocity. If the radius of the fly wheel is 50 cm, find the linear velocity of a point on its circumference.

33. The angular velocity of a particle moving in a circle of radius 50 cm is increased in 5 minutes from 100 revolutions per minute. Find (a) angular acceleration (b) linear acceleration.

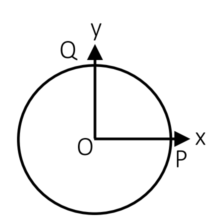
34. A body is moving in a circle of radius 100 cm with a time period of 2 second. Find the acceleration.

35. An insect trapped in a circular groove of radius 12 cm moves along the grove steadily and completes 7 revolutions in 100 s. (i) What is the angular speed, and the linear speed of the motion? (ii) Is the acceleration vector a constant vector? What is its magnitude?

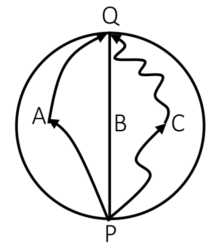
36. Calculate the centripetal acceleration of a point on the equator of earth due to the rotation of earth due to the rotation of earth about its own axis. Radius of earth = 6400 km.

37. A cyclist is riding with a speed of 27 km h^{-1}. As he approaches a circular turn on the road of radius 80.0 m, he applies brakes and reduces his speed at a constant rate of 0.5 ms^{-1} per second. Find the magnitude of the net acceleration of the cyclist.

38. A particle moves in a circle of radius 4.0 cm clockwise at constant speed of 2 cms^{-1} . If $\hat{x}$ and $\hat{y}$ are unit acceleration vectors along X-axis and Y-axis respectively, find the acceleration of the particle at the instant half way between PQ figure.



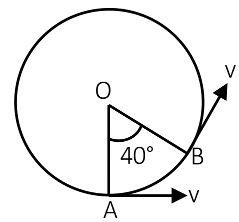
39. Three girls skating on a circular ice ground of radius 200 m start from a point P on the edge of the ground and reach a point Q diametrically opposite to P following different paths as shown in figure. What is the magnitude of the displacement vector for each? For which girl is this equal to the actual length of path skated?



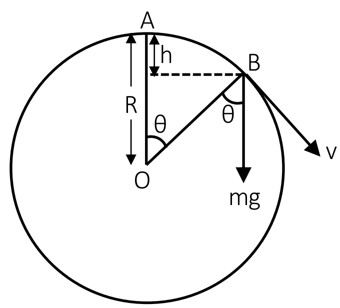
40. A cyclist starts from the centre O of a circular park of radius 1 m reaches the edge P of the park, then cycles along the circumference, and returns to the centre along QO. If the round trip takes 10 minutes, what is the (i) net displacement (ii) average velocity and (iii) average speed of the cyclist?

41. A cyclist is riding with a speed of 36 km h^{-1} . As he approaches a circular turn on the road of radius 140 m, he applies break and reduces his speed at the constant rate of 1 ms^{-2}. What is the magnitude and direction of the net acceleration of the cyclist on the circular turn?

42. A particle is moving in a circle of radius r centres at O with constant speed v . What is the change in velocity in moving from A to B? In the figure. Given ∠AOB = 40°.



43. A particle originally at rest at the highest point of a smooth vertical circle of radius R, is slightly displaced. Find the vertical distance below the highest point where the particle will leave the circle.



PART II - OBJECTIVE QUESTIONS

Projectile Motion

Single Correct Choice Type

1. A particle is projected with a certain velocity at an angle θ above the horizontal from the foot of a given plane inclined at an angle of 45° to the horizontal. If the particle strikes the plane normally, then equals

(A) $\tan^{-1}(1/3)$ (B) $\tan^{-1}(1/2)$ (C) $\tan^{-1}(1/\sqrt{3})$ (D) $\tan^{-1}(3)$

2. Two projectiles A and B are thrown with the same such that A makes angle θ with the horizontal and B makes angle θ with the vertical, then

(A) Both must have same time of fight
(B) Both must achieve same maximum height
(C) A must have more horizontal range than B
(D) Both may have same time of flight

3. A projectile is fired with a speed u at an angle θ with the horizontal. Its speed when its direction of motion makes an angle 'α' with the horizontal is

(A) usecθcosα (B) usecθsinα (C) ucosθsecα (D) usinθsecα

4. A ball is projectile from top of tower with a velocity of 5 ms^{-1} at an angle of 53° to horizontal. Its speed when it is at a height of 0.45m from the point of projection is:

(A) 2 ms^{-1} (B) 3 ms^{-1} (C) 4 ms^{-1} (D) Data insufficient

5. A particle is dropped from the height of 20 m from horizontal ground. A constant force act on the particle in horizontal direction due to which horizontal acceleration of the particle becomes 6 ms^{-2}. Find the horizontal displacement of the particle till it reaches ground.

(A) 6 m (B) 10 m (C) 12 m (D) 24 m

6. Find time of flight of projectile thrown horizontally with speed 10 ms^{-1} from a long-inclined plane which makes an angle of θ = 45° from horizontal.

(A) $\sqrt{2}$ s (B) $2\sqrt{2}$ s (C) 2 s (D) None of these

7. A projectile is fired with a velocity at right angles to the slope which is inclined at an angle θ with the horizontal. The expression for the range R along the incline is

(A) $\frac{2v^2}{g}\sec\theta$ (B) $\frac{2v^2}{g}\tan\theta$ (C) $\frac{2v^2}{g}\tan\theta\sec\theta$ (D) $\frac{2v^2}{g}\tan^2\theta$

8. A hunter tries to hunt a monkey with a small, very poisonous arrow, blown from a pipe with initial speed v_0 . The monkey is hanging on a branch of a tree at height H above the ground. The hunter is at a distance L from the bottom of the tree. The monkey sees the arrow leaving the blow pipe and immediately lose the grip on the tree, falling freely down with zero initial velocity. The minimum initial speed v_0 of the arrow for hunter to succeed while monkey is in air

(A) $\sqrt{\frac{g(H^2+L^2)}{2H}}$ (B) $\sqrt{\frac{gH^2}{H^2+L^2}}$ (C) $\sqrt{\frac{g(H^2+L^2)}{H}}$ (D) $\sqrt{\frac{2gH^2}{H^2+L^2}}$

9. A swimmer swims in still water at a speed = 5 km h^{-1}. He enters a 200 m wide river, having river flow speed = 4 km h^{-1} at point A and proceeds to swim at an angle of 127° with the river flow direction. Another point B is located directly across A on the other side. The swimmer lands on the other bank at a point C, from which he walks the distance CB with a speed = 3 km h^{-1}. The total time in which he reaches from A to B is

(A) 5 minutes (B) 4 minutes (C) 3 minutes (D) None

10. A boat having a speed of 5 km h^{-1}. in still water, crossed a river of width 1 km along the shortest possible path in 15 minutes. The speed of the river in km h^{-1} is

(A) 1 (B) 3 (C) 4 (D) $\sqrt{41}$

11. A motor boat is to reach at a point 30° upstream (w.r.t. normal) on other side of a river flowing with velocity 5 ms^{-1}. Velocity of motorboat w.r.t. water is $5\sqrt{3}$ ms^{-1}. The driver should steer the boat at an angle

(A) 120° w.r.t. stream direction
(B) 30° w.r.t. normal to the bank
(C) 30° w.r.t. the line of destination from starting point
(D) None of these

12. A flag is mounted on a car moving due north with velocity of 20 km h^{-1}. Strong winds are blowing due East with velocity of 20 km h^{-1}. The flag will point in direction

(A) East (B) North-East (C) South-East (D) South-West

13. Three ships A, B & C are in motion. The motion of A as seen by B is with speed v towards north-east. The motion of B as seen by C is with speed v towards the north-west. Then as seen by A, C will be moving towards

(A) North (B) South (C) East (D) West

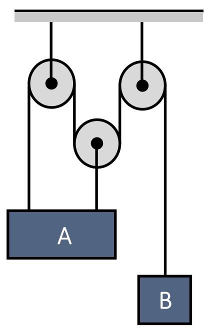
14. Wind is blowing in the north direction at speed of 2 ms^{-1} which causes the rain to fall at some angle with the vertical. With what velocity should a cyclist drive so that the rain appears vertical to him?

(A) 2 ms^{-1} South (B) 2 ms^{-1} North (C) 4 ms^{-1} West (D) 4 ms^{-1} South

15. When the driver of a car A sees a car B moving towards his car and at a distance 30 m, takes a left turn of 30° . At the same instant the driver of the car B takes a turn to his right at an angle 60°. The two cars collides after two seconds, then the velocity (in ms^{-1}) of the car A and B respectively will be: [assume both cars to be moving along same line with constant speed]

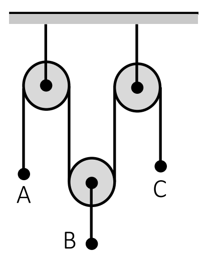
(A) 7.5, $7.5\sqrt{3}$ (B) 7.5, 7.5 (C) $7.5\sqrt{3}$, 7.5 (D) None

16. At a given instant, A is moving with velocity of 5 ms^{-1} upwards. What is velocity of B at that time



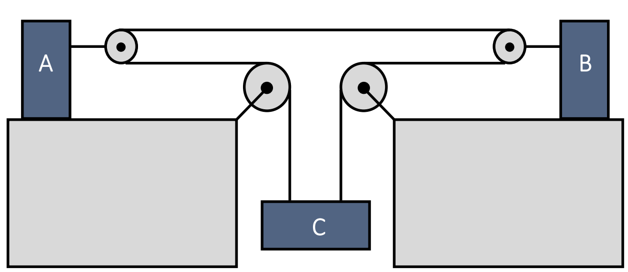
(A) 15 m/s ↓ (B) 15 m/s ↑ (C) 5 m/s ↓ (D) 5 m/s ↑

17. The pulleys in the diagram are all smooth and light. The acceleration of A is a upwards and the acceleration of C is f downwards. The acceleration of B is:



(A) 1/2(f – a) up (B) 1/2(a + f) down (C) 1/2(a + f) up (D) 1/2(a - f) up

18. If acceleration of A is 2 ms^{-2} to left and acceleration of B is 1 ms^{-2} to left, then acceleration C is:



(A) 1 ms^{-2} Upwards (B) 1 ms^{-2} Downwards (C) 2 ms^{-2} Downwards (D) 2 ms^{-2} Upward

Circular Motion

Single Correct Choice Type

19. Two racing cars of masses 1 m and 2 m are moving in circles of radius r_1 and r_2 respectively, their speeds are such that they each make a complete circle in the same time t. The ratio of the angular speed of the first to the second car is:

(A) $m_1 : m_2$ (B) $r_1 : r_2$ (C) 1 : 1 (A) $m_1r_1 : m_2 r_2$

20. A particle moves in a circle of radius 25 cm at two revolution per sec. The acceleration of the particle in ms^{-2} is:

(A) π^2 (B) $8\pi^2$ (C) $4\pi^2$ (D) $2\pi^2$

21. Two particle P and Q are located at distance r_P and r_Q respectively from the centre of a rotating disc such that $r_P > r_Q$:

(A) Both P and Q have the same acceleration
(B) Both P and Q do not have any acceleration
(C) P has greater acceleration than Q
(D) Q has greater acceleration than P

22. When particle moves in a circle with a uniform speed:

(A) Its velocity and acceleration are both constant
(B) Its velocity is constant but the acceleration changes
(C) Its acceleration is constant but the velocity changes
(D) Its velocity and acceleration both change

23. If a particle moves in a circle described equal angles in equal times, its velocity vector:

(A) Remains constant
(B) Changes in magnitude
(C) Changes in direction
(D) Changes both in magnitude and direction

24. If the equation for the displacement of a particle moving on a circular path is given by $(\theta) = 2t^3 + 0.5$, where θ is in radians and t in seconds, then the angular velocity of the particle after 2 sec from its start is:

(A) 8 rad s^{-1} (B) 12 rad s^{-1} (C) 24 rad s^{-1} (D) 36 rad s^{-1}

25. The second's hand of a watch has length 6 cm. Speed of end point and magnitude of difference of velocities at two perpendicular be:

(A) 6.28 & 0 mm s^{-1} (B) 8.88 & 4.44 mm s^{-1} (C) 8.88 & 6.28 mm s^{-1} (D) 6.28 & 8.88 mm s^{-1}

26. A fan is making 600 revolutions per minute. If after some time it makes 1200 revolutions per minute, then increases in its angular velocity is:

(A) 10π rad s^{-1} (B) 20π rad s^{-1} (C) 40π rad s^{-1} (D) 60π rad s^{-1}

27. A wheel completes 2000 revolutions to cover the 9.5 km. distance, then the diameter of the wheel is:

(A) 1.5 m (B) 1.5 cm (C) 7.5 cm (D) 7.5 m

28. A body moves with constant angular velocity on a circle. Magnitude of angular acceleration is:

(A) $r\omega^2$ (B) Constant (C) Zero (D) None of the above

29. For a particle in a uniformly accelerated (speed increasing uniformly) circular motion:

(A) Velocity is radial and acceleration is transverse only
(B) Velocity is transverse and acceleration is radial only
(C) Velocity is radial and acceleration has both radial and transverse components
(D) Velocity is transverse and acceleration has both radial and transverse components

30. A particle moves in a circular orbit under the force proportional to the distance 'r'. The speed of the particle is:

(A) Proportional of r^2 (B) Independent of r (C) Proportional to r (D) Proportional to 1/r

EXERCISE 2 - JEE ADVANCED

PART I - SUBJECTIVE QUESTIONS

Projectile Motion

1. A particle moves in the plane XY with constant acceleration a directed along the negative y axis. The equation of motion of the particle has the form $y = \alpha x - \beta x^2$, where α and β are positive constants. Find the velocity of the particle at the origin of coordinates.

2. Two seconds after projection, a projectile is moving at 30° above the horizontal, after one more second it is moving horizontally. Find the magnitude and direction of its initial velocity. [Take $g = 10\ ms^{-2}$]

3. A ball is projected from O with an initial velocity 700 cm s^{-1} in a direction 37° above the horizontal. A ball B, 500 cm away from O on the line of the initial velocity of A, is released from rest at the instant A is projected. Find:

(i) The height through which B falls, before it is hit by A
(ii) The direction of the velocity A at the time of impact

[Take $g = 10\ ms^{-2}$, sin 37° = 0.6]

4. On a frictionless horizontal surface, assumed to be the x-y plane, a small trolley A is moving along a straight line parallel to the y-axis with a constant velocity of $(\sqrt{3} - 1)\ ms^{-1}$. At a particular instant, when the line OA makes an angle 45° with the x-axis, a ball is thrown along the surface from the origin O. Its velocity makes an angle ϕ with the x-axis and it hits the trolley.

(i) The motion of the ball is observed from the frame of the trolley. Calculate the angle θ made by the velocity vector of the ball with the x-axis in this frame.
(ii) Find the speed of the ball with respect to the surface if $\phi = 4\theta/3$.

5. If R us the horizontal range and h, the greatest height of a projectile, find the initial speed. [Take $g = 10\ ms^{-2}$]

6. A stone is thrown horizontally from a tower. In 0.5 second after the stone began to move, the numerical value of its velocity was 1.5 times its initial velocity. Find the initial velocity of stone.

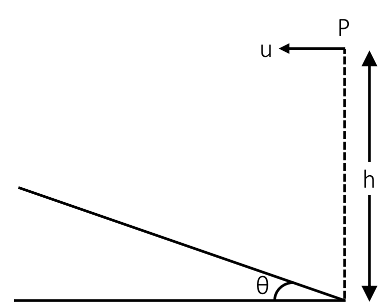
7. A shell is fired from a point O at an angle of 60° with a speed of 40 ms^{-1} & it strikes a horizontal plane through O. at a point A. The gun is fired a second time with the same angle of elevation but a different speed v. If it hits the target which starts to rise $9\sqrt{3}\ ms^{-1}$ at the same instant as the shell is fired, find v. [Take $g = 10\ ms^{-2}$]

8. A cricket ball thrown from a height of 1.8 m at an angle of 30° with the horizontal at a speed of 18 ms^{-1} is caught by another field's man at a height of 0.6 m from the ground. How far were the two men apart?

9. A batsman hits the ball at a height 4.0 ft. from the ground at projection angle of 45° and the horizontal range is 350 ft. Ball falls on left boundary line, where a 24 ft height fence is situated at a distance of 320 ft. Will the ball clear the fence?

10. (i) A particle is projected with a velocity of 29.4 ms^{-1} at an angle of 60° to the horizontal. Find the range on a plane inclined at 30° to the horizontal when projected from a point of the plane up the plane.

(ii) Determine the velocity with which a stone must be projected horizontally from a point P, so that it may hit the inclined plane perpendicularly. The inclination of the plane with the horizontal is θ and P is h metre above the foot of the incline as shown in the Figure.



11. During the volcanic eruption chunks of solid rock are blasted out of the volcano.

(i) At what initial speed would a volcanic object have to be ejected at 37° to the horizontal from the vent A at a height of 3.3 km from the ground in order to fall at B which is 9.4 m away from the base of the vent A.
(ii) What is the time of flight. [Take $g = 9.8\ ms^{-2}$]

12. A projectile is projected with an initial velocity of $(6\hat{i} + 8\hat{j})\ ms^{-1}$, $\hat{i}$ = unit vector in horizontal direction and $\hat{j}$ = unit vector in vertical upward direction then calculate its horizontal range, maximum height and time of flight.

13. An aeroplane is flying at a height of 1960 metre in a horizontal direction with a velocity of 100 ms^{-1}, when it is vertically above an object M on the ground it drops a bomb. If the bomb reaches the ground at the point N, then calculate the time taken by the bomb to reach the ground and also find the distance MN.

14. A projectile is projected from the base of a hill whose slope is that of right circular cone, whose axis is vertical. The projectile grazes the vertex and strikes the hill again at a point on the base. If θ be the semi-vertical angle of the cone, h its height u the initial velocity of projection and α the angle of projection, show that:

(i) $\tan\theta = 2\cot\alpha$ (ii) $u^2 = \frac{gh(4+\tan^2\theta)}{2}$

15. A person is standing on a truck moving with a constant velocity of 14.7 ms^{-1} on a horizontal road. The man throws a ball in such a way that it returns to the truck after the truck has moved 58.8 m. Find the speed and the angle of projection

(i) As seen from the truck
(ii) As seen from the road

16. Two bodies are thrown simultaneously from the same point. One thrown straight up and the other at an angle α with the horizontal. Both the bodies have equal velocity of v_0 Neglecting air drag, find the separation of the particle at time t.

17. Two particles move in a uniform gravitational field with an acceleration g. At the initial moment the particles were located at one point and moved with velocities 3 ms^{-1} and 4 ms^{-1} horizontally in opposite directions. Find the distance between the particles at the moment when their velocity vectors become mutually perpendicular.

18. A particle is projected from O at an elevation α and after t second it has an elevation β as seen from the point of projection. Prove that its initial velocity is:

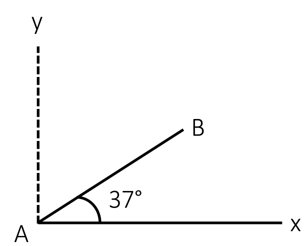
$$\frac{gt\cos\beta}{\sin(\alpha-\beta)}$$

19. The velocity of a particle when it is at its greatest height is $\sqrt{2/5}$ of its velocity when it is at half its greatest height. Find the angle of projection of the particle.

20. A man crosses a river in a boat. If he crosses the river in minimum time, he takes 10 minutes with a drift 120 m. If he crosses the river taking shortest path, he takes 12.5 minutes. Assuming $v_{b/r} > v_r$ find:

(i) Width of the river
(ii) Velocity of the boat with respect to water
(iii) Speed of the current.

21. A butterfly is flying with velocity $(10\hat{i} + 12\hat{j})$ ms^{-1} and wind is blowing along x-axis with velocity u. If butterfly starts motion from A and after some time reaches point B, find the value of u.



22. Rain is falling vertically with a speed of 20 ms^{-1} relative to air. A person is running in the rain with a velocity of 5 ms^{-1} and a wind is also blowing with a speed of 15 ms^{-1} (both towards east). Find the angle with the vertical at which the person should hold his umbrella so that he may not get drenched.

Circular Motion

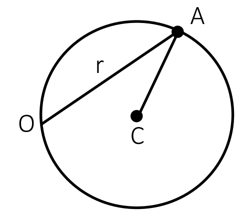
23. A bullet is moving horizontally with certain velocity. It pierces two paper discs rotating co-axially with angular speed ω separated by a distance ℓ . If the hole made by the bullet on 2nd disc is shifted through an angle θ with respect to that in the first disc, find the velocity of the bullet, (change of velocity in the bullet is neglected).

24. Position vector of a particle performing circular motion is given by $\vec{r} = 3\hat{i} + 4\hat{j}$ m, velocity vector is $\vec{v} = -4\hat{i} + 3\hat{j}$ ms^{-1}. If acceleration is $\vec{a} = -7\hat{i} - \hat{j}$ ms^{-2} find the radial and tangential components of acceleration.

25. An astronaut is rotating in a rotor having vertical axis and radius 4m. If he can withstand up to acceleration of 10g. Then what is the maximum number of permissible revolutions per second? [Take g = 10 ms^{-2}]

26. A racing-car of 1000 kg moves round a banked track at a constant speed of 108 km h^{-1} . Assuming the total reaction at the wheels is normal to the track and the horizontal radius of the track is 90 m, calculate the angle of inclination of the track to the horizontal and the reaction at the wheels.

27. A particle A moves along a circle of radius R = 50 cm so that its radius vector r relative to the point O (see figure) rotates with the constant angular velocity ω = 0.40 rad s^{-1}. Find the modulus of the velocity of the particle and modulus and direction of its total acceleration.



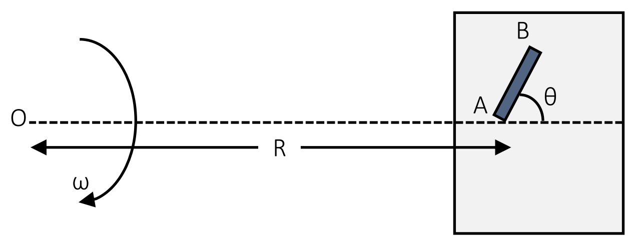
28. A wet open umbrella is held upright and is rotated about the handle at uniform rate of 21 revolutions is 44 s. If the rim of the umbrella circle is 1 meter in diameter and the height of the rim above the floor is 1.5 m, find where the drops of water spun off the rim and hit the floor.

29. A spaceman in training is rotated in a seat at the end of horizontal rotating arm of length 5 m. If he can withstand acceleration up to 9g, what is the maximum number of revolutions per second permissible? [Take g = 10 ms^{-2}]

30. An insect on the axle of a wheel observes the motion of a particle and 'find' it to take its place along the circumference of a circle of radius 'R' with a uniform angular speed ω. The axle is moving with a uniform speed 'v' relative to the ground. How will an observer on the ground describe the motion of the same point?

31. A stone is thrown horizontally with a velocity 10 ms^{-1}. Find the radius of curvature of its trajectory in 3 second after the motion began. Disregard the resistance of air.

32. A table with smooth horizontal surface is fixed in a cabin that rotates with a uniform angular velocity ω = 220 rad s^{-1} in a circular path of radius R = 700 m. A smooth groove AB of length L = 7 m is made on the surface of the table. The groove makes an angle θ = 30° with the radius OA of the circle in which the cabin rotates. A small particle is kept at the point A in the groove and is released to move along AB. Find the time taken by the particle to reach the point B.



33. A smooth sphere of radius R is made to translate in a straight line with a constant acceleration a. A particle kept on the top of the sphere is released from there at zero velocity with respect to the sphere. Find the speed of the particle with respect to the sphere as a function of the angle θ slides.

34. If a particle is rotating in a circle of radius R with velocity at an instant v and the tangential acceleration is a. Find the net acceleration of the particle.

PART II - OBJECTIVE QUESTIONS

Projectile Motion

Single Correct Choice Type

1. A projectile of mass 1 kg is projected with a velocity of $\sqrt{20}$ ms^{-1} such that it strikes on the same level as the point of projection at a distance of $\sqrt{3}$ m. Which of the following options is incorrect?

(A) The maximum height reached by the projectile can be 0.25 m
(B) The minimum velocity during its motion can be $\sqrt{15}$ ms^{-1}
(C) The time taken for the flight can be $\sqrt{3/5}$ s
(D) Minimum kinetic energy during its motion can be 6 J

2. A particle is projected from the ground with velocity u at angle θ with horizontal. The horizontal range, maximum height and time of fight are R , H and T respectively. They are given by,

$$R = \frac{u^2 \sin 2\theta}{g},\ H = \frac{u^2 \sin^2 \theta}{2g} \text{ and } T = \frac{2u \sin \theta}{g}$$

Now keeping u as fixed, θ is varied from 30° to 60° . Then,

(A) R will first increase then decrease, H will increase and T will decrease
(B) R will first increase then decrease while H and T both will increase
(C) R will first decrease while H and T will increase
(D) R will first increase while H and T will increase

3. The trajectory of particle 1 with respect to particle 2 will be

(A) A parabola (B) A straight line (C) A vertical straight line (D) A horizontal straight line

4. If $v_1\cos\theta_1 = v_2\cos\theta_2$, then choose the incorrect statement.

(A) One particle will remain exactly below of above the other particle.
(B) The trajectory of one with respect to other will be a vertical straight line.
(C) Both will have the same range
(D) None of these

5. If $v_1\sin\theta_1 = v_2\sin\theta_2$, then choose the incorrect statement.

(A) The time of flight of both the particle will be same
(B) The maximum height attained by the particles will be same
(C) The trajectory of one with respect to another will be a horizontal straight line
(D) None of these

Multiple Correct Choice Type

6. Choose the correct alternative (s)

(A) If the greatest height to which a man can throw a stone is h, then the greatest horizontal distance up to which he can throw the stone is 2h.
(B) The angle of projection for a projectile motion whose range R is m times the maximum height is $\tan^{-1}(4/n)$
(C) The time of flight T and the horizontal range R of a projectile are connected by the equation $gT^2 = 2R \tan\theta$ where θ is the angle of projection.
(D) A ball is thrown vertically up. Another ball is thrown at an angle θ with the vertical. Both of them remain in air for the same period of time. Then the ratio of heights attained by the two balls 1:1.

7. If t is the total time of flight, h is the maximum height & R is the range for horizontal motion, the x & y co-ordinates of projectile motion and time t are related as:

(A) $y = \frac{gt^2}{2}\left(\frac{t}{T}\right)\left(1 - \frac{t}{T}\right)$ (B) $y = 4h\left(\frac{x}{R}\right)\left(1 - \frac{x}{R}\right)$ (C) $y = \frac{gt^2}{2}\left(\frac{T}{t}\right)\left(1 - \frac{T}{t}\right)$ (D) $y = 4h\left(\frac{R}{x}\right)\left(1 - \frac{R}{x}\right)$

8. A particle moves in the xy plane with a constant acceleration 'g' in the negative y-direction. Its equation of motion is $y = ax - bx^2$, where a and b are constants. Which of the following is correct?

(A) The x-components of its velocity is constant.

(B) At the origin, the y-component of its velocity is $a\sqrt{\dfrac{g}{2b}}$

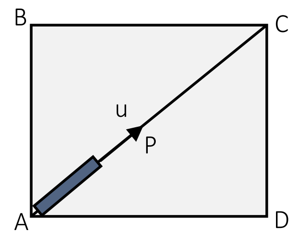
(C) At the origin, its velocity makes an angle $\tan^{-1}a$ with the x-axis.

(D) The particle moves exactly like a projectile.

9. A ball is rolled off along the edge of a horizontal table with velocity 4 ms^{-1}. It hits the ground after time 0.4 s. Which of the following are correct?

(A) The height of the table is 0.8 m
(B) It hits the ground at an angle of 60° with the vertical
(C) It covers a horizontal distance 1.6 m from the table
(D) It hits the ground with vertical velocity 4 ms^{-1}

10. A large rectangular box moves vertically downward with an acceleration a. A toy gun fixed at A and aimed towards C fires a particle P.



(A) P will hit C if a = g
(B) P will hit the roof BC, if a > g
(C) P will hit the wall CD if a < g
(D) May be either (A), (B) or (C), depending on the speed of projection of P

11. The vertical height of point P above the ground is twice that of point Q. A particle is projected downward with a speed of 5 ms^{-1} from P and at the same time another particle is projected upward with the same speed from Q. Both particles reach the ground simultaneously, if PQ is lie on same vertical line then

(A) PQ = 30 m
(B) PQ = 60 m
(C) Time of flight of the stones
(D) Time of flight of the stones = 1/3 s

12. Two particles A & B projected along different directions from the same point P on the ground with the same speed of 70 ms^{-1} in the same vertical plane. They hit the ground at the same point Q such that PQ = 480 m. Then: [Use g = 9.8 ms^{-2} , $\sin^{-1}0.96 = 74°$, $\sin^{-1}0.6 = 37°$]

(A) Ratio of their times of flights is 4:5
(B) Ratio of their maximum heights is 9:16
(C) Ratio of their minimum speeds during flight is 4:3
(D) The bisector of the angle between their directions of projection makes 45° with horizontal.

Comprehension Type

A projectile is thrown with a velocity of 50 ms^{-1} at an angle of 53° with the horizontal.

13. Choose the incorrect statement

(A) It travels vertically with a velocity of 40 ms^{-1}
(B) It travels horizontally with a velocity of 30 ms^{-1}
(C) The minimum velocity of the projectile is 30 ms^{-1}
(D) None of these

14. Determine the instants at which the projectile is at the same height.

(A) t = 1 s and t = 7 s
(B) t = 3 s and t = 5 s
(C) t = 2 s and t = 6 s
(D) All the above

15. The equation of the trajectory is given by

(A) $180y = 240x - x^2$
(B) $180y = x^2 - 240x$
(C) $180y = 135x - x^2$
(D) $180y = x^2 - 135x$

Match the Columns

Two projectiles are thrown simultaneously in the same plane from the same point. If their velocities are v_1 and v_2 at angles θ_1 and θ_2 respectively from the horizontal, then answer the following questions

16. Match the quantities is column I with possible options from column II.

Particle's Motion	Trajectory
(A) Constant velocity	(p) straight line
(B) Constant speed	(q) Circular
(C) Variable acceleration	(r) Parabolic
(D) Constant acceleration	(s) Elliptical

Assertion Reasoning Type

(A) Statement-I is true, statement-II is true and statement-II is correct explanation for statement-I
(B) Statement-I is true, statement-II is true and statement-II is NOT the correct explanation for statement-I
(C) Statement-I is true, statement-II is false.
(D) Statement-I is false, statement-II is true.

17. **Statement-I:** The speed of a projectile is minimum at the highest point.
Statement-II: The acceleration of projectile is constant during the entire motion.

18. **Statement-I:** Two stones are simultaneously projected from level ground from same point with same speeds but different angles with horizontal. Both stones move in same vertical plane. Then the two stones may collide in mid-air.
Statement-II: For two stones projected simultaneously from same point with same speed at different angles with horizontal, their trajectories may intersect at some point.

19. **Statement-I:** If separation between two particles does not change then their relative velocity will be zero.
Statement-II: Relative velocity is the rate of change of position of one particle with respect to another.

20. **Statement-I:** The magnitude of relative velocity of A with respect to B will be always less than v_A.
Statement-II: The relative velocity of A with respect to B is given by $v_{AB} = v_A - v_B$.

21. **Statement-I:** Three projectiles are moving in different paths in the air. Vertical component of relative velocity between any of the pair does not change with time as long as they are in air. Neglect the effect of air friction.
Statement-II: Relative acceleration between any of the pair of projectiles is zero.

Circular Motion

22. An object follows a curved path. The following quantities may remain constant during the motion-

(A) Speed
(B) Velocity
(C) Acceleration
(D) Magnitude of acceleration

23. The position vector of a particle in a circular motion about the origin sweeps out equal area in equal times-

(A) Velocity remains constant
(B) Speed remains constant
(C) Acceleration remains constant
(D) Tangential acceleration remains constant

EXERCISE 3 - PREVIOUS YEARS' PROBLEMS

PART I - JEE MAIN

1. A river is flowing from west to east at a speed of 5 m/min. A man on the south bank of the river, capable of swimming at 10m/min in still water, wants to swim across the river in the shortest time. He should swim in a direction **[AIEEE 1983]**

2. A boat which has a speed of 5 km/h in still water crosses a river of width 1 km along the shortest possible path in 15 mi. The velocity of the river water in km/h is **[AIEEE 1988]**

Assertion Reasoning Type

(A) If statement-I is true, statement-II is true: statement-II is the correct explanation for statement-I
(B) if statement-I is true, Statement-II is true: statement-II is not a correct explanation of statement-I
(C) If statement-I is true: statement-II is false
(D) If Statement-I is false: statement-II is true

3. **Statement-I:** For an observer looking out through the window of a fast-moving train, the nearby objects appear to move in the opposite direction to the train, while the distant objects appear to be stationary.

Statement-II: If the observer and the object are moving at velocities $\vec{v}_1$ and $\vec{v}_2$ respectively with reference to a laboratory frame, the velocity of the object with respect to the observer is $\vec{v}_2 - \vec{v}_1$ **[AIEEE 2008]**

4. For a particle in uniform circular motion the acceleration a at $\vec{a}$ point P (R, θ) on the circle of radius R is (here θ is measures from the x-axis) **[AIEEE 2010]**

(A) $-\frac{v^2}{R}\cos\theta\,\hat{i} + \frac{v^2}{R}\sin\theta\,\hat{j}$ (B) $-\frac{v^2}{R}\sin\theta\,\hat{i} + \frac{v^2}{R}\sin\theta\,\hat{j}$ (C) $-\frac{v^2}{R}\cos\theta\,\hat{i} - \frac{v^2}{R}\sin\theta\,\hat{j}$ (D) $\frac{v^2}{R}\hat{i} + \frac{v^2}{R}\hat{j}$

5. A boy can throw a stone up to a maximum height of 10 m. The maximum horizontal distance that the boy can throw the same stone up to will be **[AIEEE 2012]**

(A) $20\sqrt{2}$ m (B) 10 m (C) $10\sqrt{2}$ m (D) 20 m

6. Two cars of masses m_1 and m_2 are moving in circles of radii r_1 and r_2, respectively. Their speeds are such that they make complete circles in the same time t. The ratio of their centripetal acceleration is **[AIEEE 2012]**

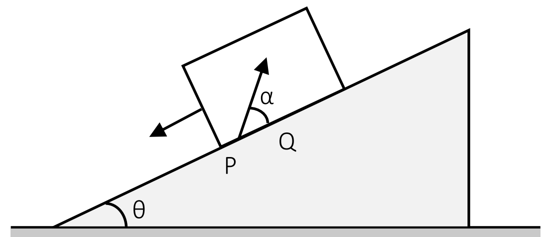
(A) $m_1r_1 : m_2r_2$ (B) $m_1 : m_2$ (C) $r_1 : r_2$ (D) 1 : 1

7. A projectile is given an initial velocity of $(\hat{i} + 2\hat{j})$ ms^{-1}, where $\hat{i}$ is along the ground and $\hat{j}$ is along the vertical. If g = 10 ms^{-2}, the equation of its trajectory is:

(A) $y = 2x - 5x^2$ (B) $4y = 2x - 5x^2$ (C) $4y = 2x - 25x^2$ (D) $y = x - 5x^2$

PART II - JEE ADVANCED

1. A large heavy box is sliding without friction down a smooth plane of inclination θ . From a point P on the bottom of the box, a particle is projected inside the box. The initial speed of the particle with respect to the box is u and the direction makes an angle α with the bottom as shown in the Figure. **[IIT JEE 1998]**

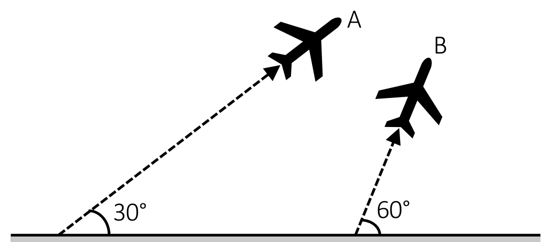


(i) Find the distance along the bottom of the box between the point of projection P and the point Q where the particle lands (Assume that the particle does not hit any other surface of the box. Neglect air resistance).

(ii) If the horizontal displacement of the particle as seen by an observer on the ground is zero, find the speed of the box with respect to the ground at the instant when the particle was projected.

2. Airplanes A and B are flying with constant velocity in the same vertical plane at angles 30° and 60° with respect to the horizontal respectively as shown in the figure. The speed of A is $100\sqrt{3}$ ms^{-1}. At time t = 0 s, an observer in A finds B at a distance of 500 m. This observer sees B moving with a constant velocity perpendicular to the line of motion of A. If at t = t_0, A just escapes being hit by B, t0 in seconds is

[JEE ADVANCED 2014]



3. The distance r of the block at time t is **[JEE ADVENCED 2016]**

(A) $\frac{R}{4}(e^{2\omega t} + e^{-2\omega t})$ (B) $\frac{R}{2}\cos(2\omega t)$ (C) $\frac{R}{2}\cos(\omega t)$ (D) $\frac{R}{4}(e^{\omega t} + e^{-\omega t})$

ANSWERS

EXERCISE 1

PART I

9. 10.54 m

10. 10 s, 99.1 ms^{-1}

11. 58 ms^{-1} ; 30° 27' with horizontal ; 44.1 m below and 150 m horizontally away from the starting point

12. 2.0 s

15. $25\sqrt{3}$ m

16. $R = 4\sqrt{H_1 H_2}$

17. $Y = 2x - 5x^2$

18. 653.06 m, 7° 42' with horizontal, 2.19 s

19. 5° 43'

20. 30° 58'

21. $4u^2/5g$

22. 37 seconds

23. 1000 km h^{-1}

24. - 3.06 ms^{-2} ; 11.43 s

25. 450 m

26. 40 km h^{-1} ; 9 min

28. (i) (a) 5 km h^{-1}, (b) 5 km h^{-1} ; (ii) (a) 0, (b) 6 km h^{-1} ; (iii) (a) 1.875 km h^{-1}, (b) 5.625 km h^{-1}

29. (i) v_0 = 667 ft s^{-1} (ii) 2667 ft (iii) v_x = 534 ft s^{-1}, v_y = 560 ft s^{-1}

30. $\pi/1800$ rad s^{-1}

31. 31.4 rad s^{-1}

32. 8π rad s^{-1} ; 1257.1 cm s^{-1}

33. $\pi/30$ rad s^{-2}, 5 $\pi/3$ cm s^{-2}

34. 987.7 cm s^{-2}

35. (i) 0.44 rad s^{-1} ; 5.3 cm s^{-1} (ii) Not constant, 2.3 cm s^{-2}

36. 0.03 ms^{-2}

37. 0.86 ms^{-2}

38. $-(\hat{x} + \hat{y})/\sqrt{2}$ cm s^{-2}

39. 400 m ; Girl B

40. (i) Zero ; (ii) 0 ; (iii) 6×10^{-3} ms^{-1}

41. $\frac{5}{7}$ ms^{-2} ; $\beta = \tan^{-1}\frac{10}{7}$

42. $\Delta v = v(-0.24\hat{i} + 0.64\hat{j})$

43. h = R/3

PART II

1. D	**2.** D	**3.** C	**4.** C	**5.** C	**6.** C	**7.** C	**8.** A	**9.** B	**10.** B	**11.** C	**12.** C
13. B	**14.** B	**15.** C	**16.** A	**17.** A	**18.** A	**19.** C	**20.** C	**21.** C	**22.** D	**23.** C	**24.** C
25. D	**26.** B	**27.** A	**28.** C	**29.** D	**30.** C						

EXERCISE 2

PART I

1. $v_0 = \sqrt{(1 + \alpha^2)a/2\beta}$

2. $20\sqrt{3}$ ms^{-1}, 60°

3. 2.55 m, 27°43'

4. (i) 45° , (ii) 2 ms^{-1}

6. 4.4 ms^{-1}

7. 50 ms^{-1}

8. 30.55 m

9. Yes

10. (i) 58.8 m (ii) $\sqrt{\dfrac{2gh}{2 + \cot^2\theta}}$

11. (i) u = 340 ms^{-1} (ii) 46 s

12. 9.8 m, 3.3 m, 1.6 s

13. 20 s, 2000 m

15. (i) 19.6 ms^{-1} upward, (ii) 24.5 ms^{-1} at 53° with horizontal

16. $v_0 t\sqrt{2(1 - \sin\alpha)}$

17. 2.47 m

19. 60°

20. 200 m, 20 m min^{-1}, 12 m min^{-1}

21. 6 ms^{-1}

22. $\tan^{-1}(1/2)$

24. $\vec{a}_r = -3\hat{i} - 4\hat{j})$ m s^{-2} ; $\vec{a}_r = -4\hat{i} - 3\hat{j})$ m s^{-2}

25. $f_{max} = 5/2\pi$ rev s^{-1}

26. 45°, $\sqrt{2} \times 10^4$ N

27. $V = 40 \times 10^{-2}$ ms^{-1} ; $a = 32 \times 10^2$ ms^{-2}

28. 0.83 metre on x-axis

29. 0.675 rev s^{-1}

30. $x = R\cos\omega t + vt$, $y = R\sin\omega t$, Cycloid

31. 334 m

32. $\sqrt{\dfrac{2L}{\omega^2 R\cos\theta}}$

33. $[2R(a\sin\theta + g - g\cos\theta)]^{1/2}$

34. $\sqrt{a^2 + \left(\dfrac{v^2}{R}\right)^2}$

PART II

1. D **2.** B **3.** B **4.** C **5.** D **6.** A, B, C, D

7. A, B **8.** A, B, C, D **9.** A, C, D **10.** A, B **11.** A, C **12.** B, C, D

13. A **14.** D **15.** A **16.** A → p; B → p, q, r, s; C → p, q, r, s; D → p, r

17. B **18.** D **19.** D **20.** D **21.** A **22.** A, D

23. B, D

EXERCISE 3

PART I

1. A **2.** B **3.** B **4.** C **5.** D **6.** C **7.** A

PART II

1. (i) $\dfrac{u^2 \sin 2\alpha}{g\cos\theta}$ (ii) $\dfrac{u\cos(\alpha + \theta)}{\cos\theta}$

2. 5

3. D